Optimization Problems

MAC 2311

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Similar to the way we used our understanding of derivatives as rates of change to solve related rates problems, we can apply our understanding of maxima/minima of functions to solve certain application problems. This is called ***optimization***. In these kinds of problems there are always two important relations to identify:

- The quantity (function) to be optimized
- The constraints on the problem

Solving Optimization Problems

1. Read the problem carefully and make a sketch to **diagram** the given information.

2. Label and define the quantities in the problem – assign variable names to all relevant quantities and **explicitly** define each variable.

3. Identify and write the formula for the quantity that is to be optimized (maximized or minimized) in terms of the variables in the problem - **objective function**

4. Identify the constraints using the conditions stated in the problem - **constraint equation**

5. If necessary, use the constraint equation to write the **objective function in terms of only one variable**.

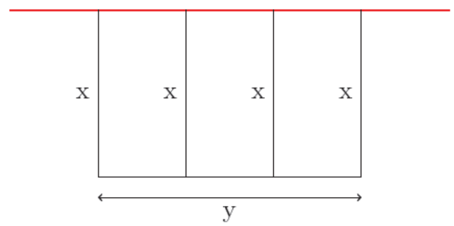
6. Find the **domain** of the objective function. (*Important! The domain determines how we verify the maximum or minimum.*)

7. Find the **absolute extreme values** on the previously found domain. (*Note: Method depends on domain.*)

8. Find requested value(s) and **summarize your results**. (*Fully answer original question and include units.*)

Example 1: Imagine that a farmer has three chickens that need to be penned up separately. The farmer wants to build three identical and adjacent rectangular pens against a barn, but only has 400-ft of fencing material to use. What dimensions of all three rectangular pens will maximize the area of all three of the enclosed pens?

1. Read the problem carefully and make a sketch to **diagram** the given information.



2. Label and define the quantities in the problem – assign variable names to all relevant quantities and **explicitly** define each variable.

$$x = \text{length of each pen}$$
$$y = \text{width of overall pens}$$
$$A = \text{area of all three pens}$$

3. Identify and write the formula for the quantity that is to be optimized (maximized/minimized) in terms of the variables in the problem - **objective function**

   This problem is asking us to maximize the overall area of the pens. So, our objective function is

$$A = xy$$

4. Identify the constraints using the conditions stated in the problem - **constraint equation**

   We are told the farmer only has 400 feet of fencing material to work with, so the amount of material used is a constraint on this problem. We can represent the amount of material used by referencing the diagram above. So, our constraint equation is:

$$4x + y = 400$$

5. If necessary, use the constraint equation to write the **objective function in terms of only one variable**.

   As of now, our objective function ($A = xy$) is in terms of two variables. We want it in terms of just one variable and we can use the constraint equation to help with that. We do this by first solving the constraint equation for just one variable:

$$\begin{aligned} 4x + y &= 400 \\ y &= 400 - 4x \end{aligned}$$

   Now, we can use this to write our objective function in terms of one variable:

$$\begin{aligned} A &= xy \\ A &= x(400 - 4x) \\ A(x) &= 400x - 4x^2 \end{aligned}$$

6. Find the **domain** of the objective function. (*Important! The domain determines how we verify the maximum or minimum.*)

   Since we are working with a real-world problem and since x and y both represent the length/width of material, then neither of them can be negative. So,

$$\begin{aligned} x \geq 0 \text{ and } y \geq 0 \rightarrow 400 - 4x &\geq 0 \\ -4x &\geq -400 \\ x &\leq 100 \end{aligned}$$

   So, we see that x has to be both greater than or equal to 0 and less than or equal to 100. So, our domain is:

$$D : 0 \leq x \leq 100 \text{ or } x \in [0, 100]$$

7. Find the **absolute extreme values** on the previously found domain. (*Note: Method depends on domain.*)

   From previous work with absolute extrema, we know to find extreme values of a function, we must first find the critical points of the function. So, let's start by finding the critical points of $A(x)$:

$$A(x) = 400x - 4x^2 \Rightarrow A'(x) = 400 - 8x$$

$\underline{A'(x) = 0}$	$\underline{A'(x) \text{ undefined}}$
$400 - 8x = 0$	$A'(x)$ is a polynomial and is defined everywhere
$-8x = -400$	
$x = 50$	

   So, $x = 50$ is our only critical point.

   Now, we find the function values at the critical point we found and the endpoints of the domain.

$$\begin{aligned} A(50) &= 400(50) - 4(50)^2 = 10,000 \\ A(0) &= 400(0) - 4(0)^2 = 0 \\ A(100) &= 400(100) - 4(100)^2 = 0 \end{aligned}$$

   Since we are interested in how to maximize the area of the pens, we want the absolute maximum of $A(x)$. From the function values we found we see:

   There is an absolute maximum of 10,000 at $x = 50$.

8. Find requested value(s) and **summarize your results**. (*Fully answer original question and include units.*)

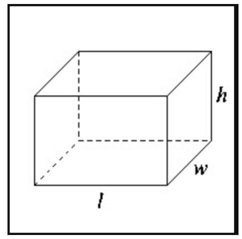
   The problem asked us to find the dimensions that would yield the maximum area. We found that the maximum area is achieved when $x = 50$, so we need to also find what that means for the variable, y:

$$x = 50 \Rightarrow y = 400 - 4x = 400 - 4(50) = 200$$

   Final answer: So, in order to maximize the area of all three pens, the length of each pen should be 50-ft and the overall width of the pens should be 200-ft.

Example 2: A rectangular container with a square base is to be made from 150-m^2 of material. Find the maximum volume such a container can have.

Diagram with labels:



Define variables:

$$\begin{aligned} w &= \text{width of container} \\ l &= \text{length of container} \\ h &= \text{height of container} \\ V &= \text{volume of container} \\ S &= \text{surface area of container} \end{aligned}$$

Objective function: What are we trying to optimize? – We are trying to maximize the volume of the container, so volume of the container is our objective function.

The container has a square base $\Rightarrow l = w$. So,

$$V = lwh = w^2h$$

Constraint equation: There is only 150-m^2 of material available to make the container. This value represents the surface area of the container. So, our constraint equation is:

$$S = 2w^2 + 4wh = 150$$

Solve constraint equation for one variable: Since the goal is to write our objective function in terms of one variable, we use our constraint equation to help with this by solving the constraint equation for one variable. Looking at our constraint, we see that algebraically it is easier for us to solve for h:

$$\begin{aligned} 2w^2 + 4wh &= 150 \\ 4wh &= 150 - 2w^2 \\ h &= \frac{150 - 2w^2}{4w} = \frac{75 - w^2}{2w} \end{aligned}$$

Objective function in terms of one variable: Now, we can substitute h in terms of w into our objective functon:

$$\begin{aligned} V &= w^2h \\ V(w) &= w^2\left(\frac{75 - w^2}{2w}\right) = \frac{75w^2 - w^4}{2w} = \frac{75}{2}w - \frac{1}{2}w^3 \end{aligned}$$

Domain: Width and height can't be negative, so:

$$\begin{aligned} w \geq 0 \text{ and } h \geq 0 \Rightarrow \frac{75 - w^2}{2w} &\geq 0 \\ 75 - w^2 &\geq 0 \\ -w^2 &\geq -75 \\ w^2 &\leq 75 \\ w &\leq \sqrt{75} \end{aligned}$$

So, the domain is $0 \leq w \leq \sqrt{75}$ or $w \in \left[0, \sqrt{75}\right]$

Find absolute extrema on domain: Find critical points and function values at critical points and endpoints of domain.

$$V'(w) = \frac{75}{2} - \frac{3}{2}w^2$$

$\underline{V'(w) = 0}$

$$\frac{75}{2} - \frac{3}{2}w^2 = 0$$
$$-\frac{3}{2}w^2 = -\frac{75}{2}$$
$$w^2 = 25$$
$$w = 5,\ w = -5$$

$\underline{V'(w) \text{ undefined}}$

$V'(w)$ is defined everywhere

Critical points: $w = 5$

We note that $w = -5$ is not on the domain and is therefore not a critical point.

Now, we can check our function values:

$$\begin{aligned} V(5) &= \frac{75}{2}(5) - \frac{1}{2}(5)^3 = 125 \\ V(0) &= \frac{75}{2}(0) - \frac{1}{2}(0)^3 = 0 \\ V\left(\sqrt{75}\right) &= \frac{75}{2}(\sqrt{75}) - \frac{1}{2}(\sqrt{75})^3 = 0 \end{aligned}$$

So, we have an absolute maximum of 125 at $w = 5$.

Summary of results: The maximum volume a rectangular container with a square base made from 150-m^2 of material can have is 125-m^3.

Example 3: Determine the side lengths to maximize the area of a rectangle with a perimeter of 100 meters.

Example 4: A right triangle whose hypotenuse is $\sqrt{48}$ inches long is revolved about one of its legs to generate a right circular cone (see figure). Find the lengths of the other two sides of the triangle that will maximize the volume of the cone and find the maximum volume that will occur.