# Antiderivatives; The Indefinite Integral

MAC 2311

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# 1 The Indefinite Integral

## 1.1 Antiderivatives

A function $F(x)$ is an **antiderivative** of $f(x)$ provided $\frac{d}{dx}[F(x)] = f(x)$ for all $x$ in the domain of $f$.

- ***If f(x) is the derivative of F(x), then F(x) is the antiderivative of f(x).***

For example, the function $F(x) = x^2$ is an antiderivative of $f(x) = 2x$ because

$$F'(x) = \frac{d}{dx}\left[x^2\right] = 2x = f(x)$$

Furthermore, the function $F(x) = x^2 + C$, where $C$ is a constant, is also an antiderivative of $f(x) = 2x$ since

$$F'(x) = \frac{d}{dx}\left[x^2 + C\right] = 2x + 0 = 2x = f(x)$$

So, each of the following are examples of antiderivatives of $2x$:

$$x^2 + 1 \qquad x^2 - 2 \qquad x^2 + \pi \qquad x^2 - \frac{1}{3}$$

**The Most General Antiderivative**

If $F(x)$ is an antiderivative of $f(x)$, then all antiderivatives of $f(x)$ have the form

$$F(x) + C$$

where $C$ is an arbitrary constant.

**Example 1:** Find the most general antiderivative of $f(x) = \cos(x)$.

The antiderivative of $\cos(x)$ will be the function we differentiate in order to obtain $\cos(x)$. From knowledge of derivatives, we know that we take the derivative of $\sin(x)$ to get $\cos(x)$. So, the antiderivative of $f(x) = \cos(x)$ is

$$F(x) = \sin(x) + C$$

## 1.2 Indefinite Integrals

Given $F(x)$ is an antiderivative of $f(x)$, then the set of all antiderivatives of $f(x)$ is the **indefinite integral** of $f(x)$ with respect to $x$. The indefinite integral is denoted by

$$\int f(x)dx = F(x) + C$$

- Read: "The integral of $f(x)$ with respect to $x$ is $F(x)$ plus a constant."
- The symbol $\int$ is an **integral** sign
- The function $f(x)$ is the **integrand** of the integral
- $x$ is the **variable of integration**
- $C$ is the **constant of integration**

$$\textbf{\textit{NOTE:}}\ \frac{d}{dx}\left[\int f(x)dx\right] = f(x)$$

**Example 2:** In the previous section, we indicated the most general antiderivatives for both $2x$ and $\cos x$. We can express these as indefinite integrals:

- $\displaystyle\int 2xdx = x^2 + C$
- $\displaystyle\int \cos(x)dx = \sin(x) + C$

## 1.3 Some Common Indefinite Integrals

- $\displaystyle\int 1dx = x + C$
- $\displaystyle\int x^n dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$
- $\displaystyle\int \sin(x)dx = -\cos(x) + C$
- $\displaystyle\int \cos(x)dx = \sin(x) + C$
- $\displaystyle\int \frac{1}{x}dx = \ln|x| + C$
- $\displaystyle\int e^x dx = e^x + C$
- $\displaystyle\int b^x dx = \frac{b^x}{\ln b} + C$
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$
- $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x + C$
- $\displaystyle\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$

***NOTE: This list is not exhaustive.**

There are two important properties of indefinite integrals that will be useful in the evaluation of integrals:

- $\displaystyle\int (f(x) \pm g(x))\,dx = \int f(x)dx \pm \int g(x)dx$
- $\displaystyle\int af(x)dx = a\int f(x)dx$, where $a$ is a constant multiple

**Example 3:** Evaluate the following:

- $\int \sqrt[3]{x}dx$

$$\begin{aligned} \int \sqrt[3]{x}dx &= \int x^{1/3}dx \\ &= \frac{1}{\frac{1}{3}+1}x^{\frac{1}{3}+1} + C \\ &= \frac{1}{\frac{4}{3}}x^{4/3} + C \\ &= \frac{3}{4}x^{4/3} + C \end{aligned}$$

*We can check our answer by taking the derivative of our solution to make sure we obtain the original integrand.*

$$\frac{d}{dx}\left[\frac{3}{4}x^{4/3} + C\right] = \frac{3}{4} \cdot \frac{4}{3}x^{(4/3)-1} + 0 = x^{1/3} = \sqrt[3]{x}$$

- $\int \left(8 - x^5 + \frac{4}{x}\right) dx$

- $\int \left(\frac{5}{1+x^2} + e^x - \frac{2}{x^2}\right) dx$

- $\int \sin(3x)dx$

## 1.4 Differential Equations and Initial Value Problems

A **differential equation** is an equation that involves a derivative of an unknown function. So, we can know the derivative of a function, but not the function itself. (*e.g.* We are given $f'(x)$, but not $f(x)$.) A number of equations that we work with in science and engineering are derived from a specific type of differential equation called an **initial value problem**. Finding a function, $f(x)$, given the derivative of the function, $f'(x)$, and a function value, $f(a) = b$, is an initial value problem.

**Solving an Initial Value Problem**

Given $f'(x)$ and $f(a) = b$, we find an exact solution for $f(x)$ by:

1. Find the most general antiderivative of $f'(x)$: $f(x) = \displaystyle\int f'(x)dx$
2. Using the initial value, $f(a) = b$, plug into the antiderivative and solve for $C$.

**Example 4:** Solve the initial value problem: $f'(x) = 7x^6 - 4x^3 + 12$; $f(1) = 25$

So, given $f'(x)$ and a function value, we're trying to find an exact solution for $f(x)$. We start by finding the most general antiderivative of $f'(x)$:

$$f(x) = \int f'(x)dx \quad = \quad \int \left(7x^6 - 4x^3 + 12\right) dx = x^7 - x^4 + 12x + C$$

Now, we need to solve for $C$. We do this by using the initial value given, $f(1) = 25$:

$$\begin{aligned} f(1) = 1^7 - 1^4 + 12(1) + C &= 25 \\ 1 - 1 + 12 + C &= 25 \\ C &= 13 \end{aligned}$$

So, our exact solution for $f(x)$ is:

$$f(x) = x^7 - x^4 + 12x + 13$$

**Example 5:** Solve the following initial value problems.

- $f'(x) = 4e^x$; $f(0) = 1$
- $f''(x) = 2\cos x + \sin x$; $f(0) = 0$; $f'(0) = 4$