# The Substitution Rule

MAC 2311

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## 1 Indefinite Integrals Review

The set of all antiderivatives of a function is the indefinite integral of that function. If $f'(x)$ is the derivative of $f(x)$, then $f(x)$ is the antiderivative of $f'(x)$.

If

$$\frac{d}{dx}[f(x)] = f'(x)$$

then

$$\int f'(x)dx = f(x) + C$$

**Examples:** Evaluate the following.

- $\int x^8 dx$

- $\int (\cos x - \sec x \tan x)\, dx$

- $\int \left(5e^x - \frac{3}{x}\right) dx$

- $\int \left(\frac{1}{\sqrt{1-x^2}} + 4\right) dx$

# 2 The Substitution Rule

## 2.1 The Chain Rule Revisited

Let's recall the chain rule: For a composite function, $f(g(x))$,

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))g'(x)$$

So, by the rule for antiderivatives,

$$\int f'(g(x))g'(x)dx = f(g(x)) + C$$

Remember, a key step in the chain rule was to identify the "inner function". So, in the case of $f(g(x))$, we need to identify the function that is $g(x)$. When presented with a "complex" integral we need to evaluate, we can take this same approach of identifying an inner function in order to change integral into a simpler one to evaluate in terms of a new variable.

**The Substitution Rule**

Given $f'(g(x))g'(x)$, if we let $u = g(x)$, then

$$\begin{aligned} \int f'(g(x))g'(x)dx &= \int f'(u)du \\ &= f(u) + C \\ &= f(g(x)) + C \end{aligned}$$

**NOTE:** If $u = g(x)$, then $\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)dx$

***Steps for the Substitution Rule***

1. Let the new variable, $u$, be some function of $x$ appearing in the integrand.
2. Take the derivative of $u$ with respect to $x$.
   - Use this to solve for $du$ in terms of $x$ and $dx$.
3. Replace all the $x$-terms in the integral with corresponding $u$'s and $du$.
4. Evaluate this (simpler) integral in terms of $u$.
5. Replace all the $u$'s in the final answer back in terms of $x$. (Don't forget the constant of integration for indefinite integrals.)

***NOTE:***

- ***Your first choice of u may not work out. If this happens, you should try another choice of u.***
- ***Never let $u = x$.***

## 2.2 Perfect Substitution

**Example:** Evaluate $\int 2x\sqrt{x^2-9}\ dx$

Let's start by identifying an appropriate choice of $u$. We should choose $u$ to be a function of $x$ in the integrand that can be considered an "inner function". So, a good choice of $u$ would be:

$$u = x^2 - 9$$

We take the derivative of $u$ with respect to $x$ so that we can find $du$ in terms of $x$ and $dx$:

$$\begin{aligned}
\frac{du}{dx} = \frac{d}{dx}[u] &= \frac{d}{dx}\left[x^2-9\right] = 2x \\
\frac{du}{dx} &= 2x \\
du &= 2xdx
\end{aligned}$$

Now, we can replace all the $x$ terms in the integral with the corresponding $u$ terms, evaluate the integral in terms of $u$, and then put everything back in terms of $x$.

$$\begin{aligned}
\int 2x\sqrt{x^2-9}\ dx &= \int \sqrt{x^2-9}\ 2xdx \leftarrow \textit{rearranging integrand to make the substitution more apparent} \\
&= \int \sqrt{u}\ du \\
&= \int u^{1/2}\ du \\
&= \frac{2}{3}u^{3/2} + C \\
&= \frac{2}{3}\left(x^2-9\right)^{3/2} + C \leftarrow \textit{Finish by replacing u back in terms of x}
\end{aligned}$$

**Example:** Evaluate $\int 4\cos(4x)\ dx$

## 2.3 Substitution by Introducing a Constant

**Example:** Evaluate $\int x \sin\left(2x^2\right) dx$

Let's start by identifying an appropriate $u$ and finding $du$:

$$\begin{aligned} u &= 2x^2 \\ \frac{du}{dx} = 4x \Rightarrow du &= 4xdx \end{aligned}$$

We see that $du$ gives us a constant multiple of 4 that is not present in our original integrand. Since we only need the "$xdx$" portion of $du$, we can introduce a constant to produce this:

$$\begin{aligned} &du = 4xdx \\ &\frac{1}{4}du = xdx \end{aligned}$$

We are now able to rewrite our integral in terms of $u$ and evaluate:

$$\begin{aligned} \int x \sin\left(2x^2\right) dx &= \int \sin\left(2x^2\right) xdx \\ &= \int \sin(u) \cdot \frac{1}{4}du \\ &= \frac{1}{4}\int \sin(u)du \\ &= \frac{1}{4} \cdot -\cos(u) + C \\ &= -\frac{1}{4}\cos\left(2x^2\right) + C \end{aligned}$$

**Example:** Evaluate $\int e^{-7x}dx$

## 2.4 Not So Apparent Substitutions

**Example:** Evaluate $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$

At first glance, it may be tempting to let $u = 1 - e^{2x}$. However, if we were to find $du$ with this choice of $u$, we would obtain $du = -2e^{2x}dx$. This expression (or any variation of it) does not appear in our integrand, so this is not an appropriate choice of $u$.

Instead, let's first rewrite the expression in the integrand:

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}dx$$

From here we can identify another "inner function", $e^x$, and we can try this as a choice of $u$:

$$\begin{aligned} u &= e^x \\ \frac{du}{dx} = e^x \Rightarrow du &= e^x dx \end{aligned}$$

Now, we are able to rewrite the integral in terms of $u$ and evaluate:

$$\begin{aligned} \int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{e^x}{\sqrt{1-(e^x)^2}}dx &= \int \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x dx \\ &= \int \frac{1}{\sqrt{1-u^2}}du \\ &= \sin^{-1}(u) + C \\ &= \sin^{-1}(e^x) + C \end{aligned}$$

**Example:** Evaluate $\displaystyle\int \cot(x)dx$

## 2.5 Substitution Involving Manipulating Variables

**Example:** Evaluate $\int x\sqrt[3]{x+2}\ dx$

An intuitive choice of $u$ is $x+2$, so:

$$\begin{aligned} u &= x+2 \\ \frac{du}{dx} = 1 \Rightarrow du &= dx \end{aligned}$$

Our goal in using substitution to evaluate integrals is to be able to rewrite the entire integrand in terms of $u$. With our choice of $u$ and $du$, we can rewrite the $x+2$ and $dx$ parts of the integrand, but we need to also be able to rewrite the $x$ in terms of $u$. Using algebra, we can use our choice of $u$ to solve for $x$ in terms of $u$:

$$\begin{gathered} u = x+2 \\ x = u-2 \end{gathered}$$

Now, we can write the given integral in terms of $u$ and evaluate:

$$\begin{aligned} \int x\sqrt[3]{x+2}\ dx &= \int (u-2)\sqrt[3]{u}\ du \\ &= \int (u-2)u^{1/3}\ du \\ &= \int \left(u^{4/3} - 2u^{1/3}\right) du \\ &= \frac{3}{7}u^{7/3} - \frac{3}{2}u^{4/3} + C \\ &= \frac{3}{7}(x+2)^{7/3} - \frac{3}{2}(x+2)^{4/3} + C \end{aligned}$$

**Example:** Evaluate $\int \frac{3x^5}{x^3+5}dx$