Area Under a Curve and Definite Integrals; The Fundamental Theorem of Calculus

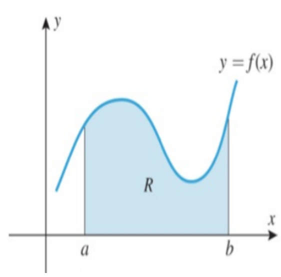
MAC 2311

Florida International University

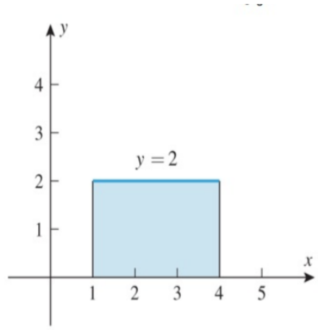
1 Definite Integrals

1.1 Area Under a Curve

Suppose we want to determine the area of a region between a function's curve and the x-axis on an interval from $[a, b]$. For example, in the figure below, if we want to find the area of the shaded region, R.



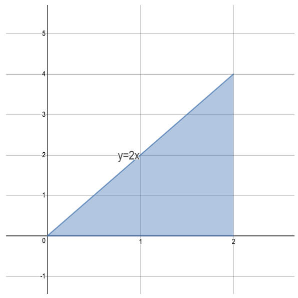
If the shape of a curve is a common one, finding this area can be done by using geometric formulas. For example, let's say we wanted to find the area of the region between the curve of $y = 2$ and the x-axis from $x = 1$ to $x = 4$.

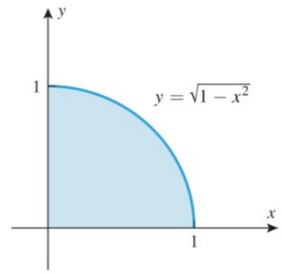


We see the region between the x-axis and the curve on the interval makes a rectangular shape. So, we can find the area of the region by using the formula for the area of a rectangle.

$$\boldsymbol{Area} = \boldsymbol{length} \cdot \boldsymbol{width} = 3 \cdot 2 = 6$$

Example: Find the areas of the shaded regions between the curves and the x-axis.

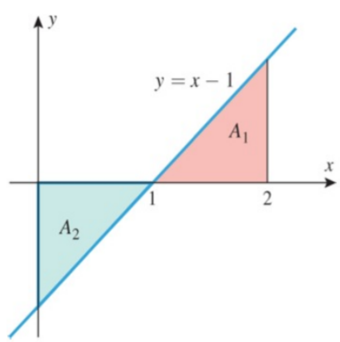




1.2 Net Area

Now, suppose we're interested in the net area of the region between a curve and the x-axis on an interval that causes the curve to go above *and* below the x-axis. In this case we are looking for the **net area**. To find this we treat the ***area below the x-axis as negative and the area above the x-axis as positive***.

For example, suppose we wanted to find the area of the region between the curve $y = x - 1$ and the x-axis on the interval $[0, 2]$.



The region between the x-axis and the curve on the given interval gives two triangular regions, one below the x-axis (A_2) and one above (A_1). So, to find the ***net area*** we would take the sum of the areas of the regions, treating the one above the x-axis as positive and the one below as negative.

$$\begin{aligned} \textbf{\textit{Net Area}} &= A_1 + (-A_2) \\ &= A_1 - A_2 \\ &= \frac{1}{2}(1)(1) - \frac{1}{2}(1)(1) \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0 \end{aligned}$$

1.3 Net Area as an Integral

So far we have only considered curves that give us "nice" shapes to work with. But what happens when we do not have a specific formula for finding the areas of these regions between the curve and the x-axis? This is where the ***definite integral*** helps us.

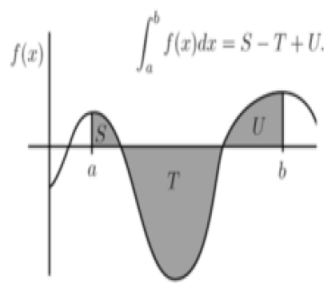
The Definite Integral

If a function $f(x)$ is continuous on an interval $[a, b]$, then the net area between the graph of $f(x)$ and the x-axis is

$$\int_a^b f(x)dx$$

Note: a is the lower limit of integration and b is the upper limit of integration.

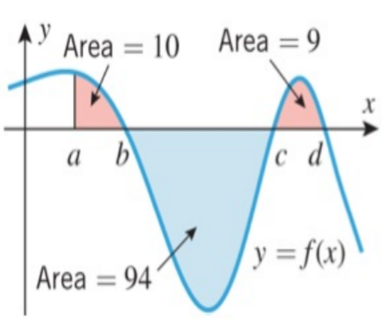
So, the definite integral gives us the net area of the region between the curve of a function and the x-axis.



Properties of Definite Integrals

- $\int_a^b kf(x)dx = k\int_a^b f(x)dx$, k is a constant
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$

Example: Use the given graph to evaluate the integrals.



- $\int_a^b f(x)dx$
- $\int_b^c \frac{1}{2}f(x)dx$
- $\int_d^c f(x)dx$
- $\int_a^c f(x)dx$

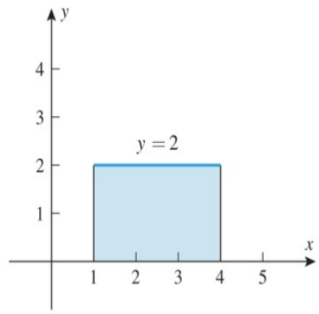
2 The Fundamental Theorem of Calculus

<u>**The Fundamental Theorem of Calculus**</u>

If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is an antiderivative of $f(x)$, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Example: Compute the net area between the function $f(x) = 2$ and the x-axis on the interval $[1, 4]$

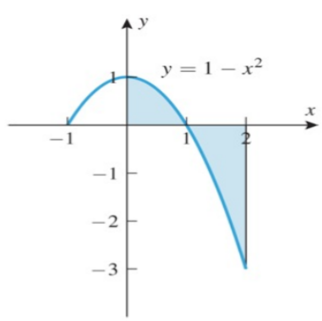


Using the Fundamental Theorem of Calculus, we can compute the net area of this region by using a definite integral.

$$\textbf{\textit{Net Area}} = \int_1^4 2dx = 2x\Big|_1^4 = 2(4) - 2(1) = 8 - 2 = 6$$

Recall: We found this same net area in the first example by using a geometric formula.

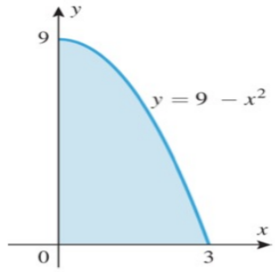
Example: Compute the net area between the function $f(x) = 1 - x^2$ and the x-axis on the interval $[0, 2]$.



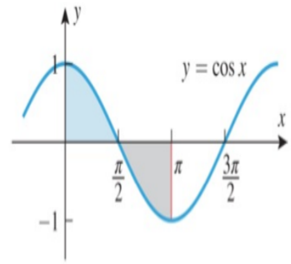
$$\textbf{\textit{Net Area}} = \int_0^2 \left(1 - x^2\right) dx = \left[x - \frac{1}{3}x^3\right]\Big|_0^2 = \left[2 - \frac{8}{3}\right] - [0 - 0] = -\frac{2}{3}$$

Example: Compute the net area of the following functions on the given intervals.

- $f(x) = 9 - x^2$; $[0, 3]$



- $f(x) = \cos x$; $[0, \pi]$



<u>**The Fundamental Theorem of Calculus - Substitution Rule**</u>

If $u = g(x)$, where $g'(x)$ is continuous on $[a, b]$, and f is continuous on the range of g, then on that interval:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Note: If $u = g(x)$, then $du = g'(x)dx$; $x = a \Rightarrow u = g(a)$ and $x = b \Rightarrow u = g(b)$.

Example: Evaluate $\displaystyle\int_1^{\sqrt{2}} 2x\sqrt{x^2-1}dx$

$$u = x^2 - 1 \qquad x = 1 \to u = 1^2 - 1 = 0$$
$$du = 2xdx \qquad x = \sqrt{2} \to u = \left(\sqrt{2}\right)^2 - 1 = 1$$

$$\begin{aligned}\int_1^{\sqrt{2}} 2x\sqrt{x^2-1}dx = \int_0^1 \sqrt{u}du &= \int_0^1 u^{1/2}du \\ &= \frac{2}{3}u^{3/2}\Big|_0^1 \\ &= \frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2} \\ &= \frac{2}{3} - 0 \\ &= \frac{2}{3}\end{aligned}$$

****Note: When using the substitution rule with a definite integral, don't forget to change the limits of integration. Also, there's no need to go back to terms of x.***

Examples: Evaluate the following definite integrals.

- $\int_{2}^{5} (2x+1)dx$

- $\int_{-1}^{2} \frac{x}{x^2-5}dx$

- $\int_{0}^{1} \frac{1}{1+x^2}dx$

- $\int_{1}^{2} x\sqrt{x-1}dx$