The Mean Value Theorem

MAC 2311

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When evaluating limits, we saw that **indeterminate forms** can be evaluated using the following result.

Theorem (L'Hôpital's Rule)**.** *Let $f'(x)$ and $g'(x)$ be continuous at $x = a$, $f(a) = g(a) = 0$, and $g'(a) \neq 0$. If*

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

has the property that either

$$\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \to \frac{0}{0} \quad \textit{\textbf{or}} \quad \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \to \frac{\pm\infty}{\pm\infty}$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

This works because we assume $g'(a) \neq 0$. If we want to do it in worse cases, we need a new tool called **the Mean Value Theorem** to make sure that our functions and their derivatives don't change (remember - derivatives measure change) unpredictably.

We can state the problem in a different context, namely when we try to predict the behavior of a function using its derivative. The main result we would like to pin down is

Theorem. *If $f'(x) > 0$ for all values x in an interval (a, b), then f is increasing on that interval.*

In other words, for any p and q in (a, b) with $p < q$, $f(p) < f(q)$, and this *theorem* means we can tell that is happening just by looking at the sign of the derivative.

If for some p in (a, b), $0 < f'(p)$. Then

$$0 < f'(p) \implies f'(p)(x-p) = f(x) - f(p) = \frac{f(x) - f(p)}{x - p} \implies 0 < f(x) - f(p)$$

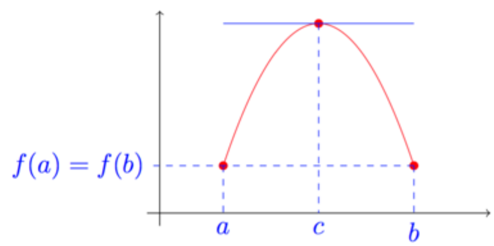
This would mean that if $x > p$, we'd get that $f(x) > f(p)$, but how do we know that this works for any x we choose? Do we need to do this infinitely many times for all the values between p and b? No, but we need a strategy.

To see how to do this, let's start with a warm-up case. This is called Rolle's Theorem.

Theorem (Rolle's Theorem)**.** *Suppose that $f(x)$ is continuous on an interval $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Then, there is at least one point c in (a, b) where $f'(c) = 0$. That is, there is at least one interior point c in the domain of f such that the graph of $f(x)$ has a horizontal tangent line.*

Proof. We first have to show that $f(x)$ has a maximum or minumum somewhere between a and b. The Extreme Value Theorem gives us that. Then either f is zero everywhere, or there is some point where $f(x) \neq 0$, say $x = c$. By the EVT, $f'(c) = 0$. □

Graphically, Rolle's Theorem is presented below.



The Mean Value Theorem is a little less obvious, but it is a generalization of Rolle's Theorem.

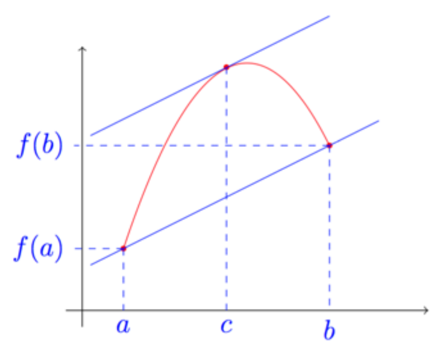
Theorem (Mean Value Theorem)**.** *Suppose that $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interior interval (a, b). Then, there is at least one point c in (a, b) such that*

$$f'(c) = \frac{(f(b) - f(a))}{(b - a)}$$

In other words, there is a tangent line at some number $x = c$ between a and b that has the same slope as the secant line between the points $(a, f(a))$ and $(b, f(b))$.

Proof. Define a new function $g(x)$ using the given function so that $g(a) = g(b) = 0$, then apply Rolle's theorem. □

Graphically, the Mean Value Theorem is presented below.



For our purposes, the most important aspects of the Mean Value Theorem are its consequences:

- If the derivative of a function is zero, then the function must be a constant.

  We've often assumed this is true, because we already learned about the reverse of this statement: the derivative of any constant is zero. Try to prove this yourself using MVT.

- If f is continuous, differentiable, and has a positive derivative, then f is an increasing function. If f has a negative derivative, then it is a decreasing function.

We can now prove this conclusively. Use the existence of a value c in the interval (a, b) to show that the positive (or negative) derivative at that point forces the function to increase from $x = p$ to $x = q$ (or any pair of values in (a, b).

- If two functions have the same derivative (i.e., $f'(x) = g'(x)$), then $f(x)$ and $g(x)$ must only differ by a constant: $f(x) = g(x) + C$.

  This is new and important for other reasons.

Example 1: For $f(x) = 3x^2 + 2x + 5$ on $[-1, 1]$, determine whether MVT applies. If so, find the points that are guaranteed to exist by MVT.