Parametric Curves

MAC 2311

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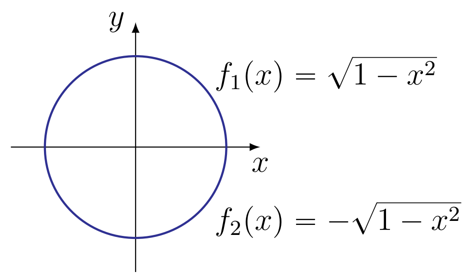
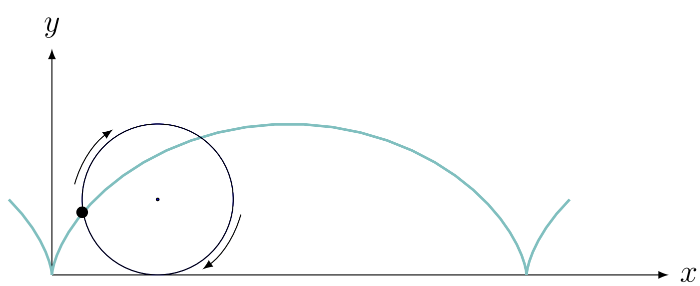
Up to this point, the use of an equation relating two variables was the only alternative to represent figures that cannot be expressed as a single, real-valued function. For instance, to represent the unit circle in the plane, we needed to used the two functions ($f_1(x)$ and $f_2(x)$) indicated in the figure below:



The equation of the unit circle (i.e. $x^2+y^2=1$) proved instead to be a much simpler representation when finding tangent lines to the circle. In this case we developed the technique of implicit differentiation.

In problems that include other variables, such the time, an alternative to these representation of curves might also help easing calculations. This can be illustrated by a point fixed in the rim of a rotating wheel. If the center of this rotating wheel is fixed (as in a pulley), the trajectory of this point can be represented by the unit circle equation. If this wheel, on the other hand, is rolling freely on a flat surface (as if it were to be part of a bicycle), the trajectory of this point would look more like the figure below:

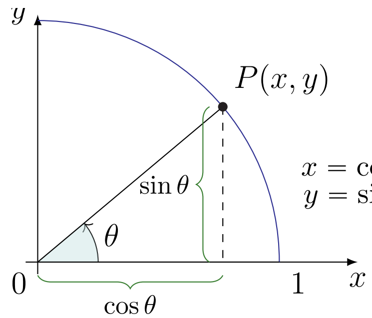


The representation of this curve, called a cycloid, is much more complex using functions[1] or an equation. A more convenient alternative is inspired by the relationship between the unit circle and the trigonometric functions $\sin(\theta)$ and $\cos(\theta)$. The main idea is to think of the angle, θ, that the line from the center of the circle at the origin $(0, 0$ to a point on the circle $P(x, y)$ makes with the positive x-axis.

[1]using y as the independent variable, the cycloid curve of radius 1 can be represented by the function

$$x = cos^{-1}(1-y) - \sqrt{y(2-y)}$$

This angle leads to define, as illustrated in the figure below, both coordinates of the point P as functions of θ.



Using the right values for the angle θ, the circle then can also be represented by the following conditions:

$$x = \cos\theta,\ y = \sin\theta,\ \text{for } 0 \leqslant \theta \leqslant 2\pi$$

The first two equations are called **parametric equations**, and θ is called a parameter, and all together are known as the parametric representation of the curve, or simply **parametric curve**. Since most problems where parametric curves are useful involve time as a parameter, t is used instead of θ, so that parametric curves in general are defined as follows,

$$x = f(t),\ y = g(t),\ \text{for } a \leqslant t \leqslant b$$

Example 1: Find a parametric representation of the circle of radius 9 centered at the origin. Is it possible to find more than one parametric representation of this circle?

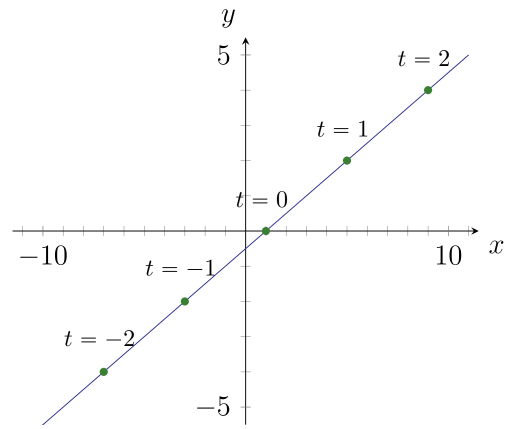
The previous example illustrates that curves can have more than one parametrization, depending for example on how the parameter varies. Moreover, if curves are thought to be drawn as the parameter changes, these changes can represent different **curve orientations**. If the parameter increases as the curve follows its path, this orientation is called positive. For instance, in the unit circle parametrization found above, a positive orientation corresponds to a counterclockwise direction.

Although recognizing the curve based on its parametrization is not initially intuitive, its shape and orientation can be sketched using a sample of values for its parameter. Consider, for example, the parametrization

$$x = 1 + 4t,\ y = 2t,\ \text{for } -\infty < t < \infty.$$

By substituting $t = 0$, one quickly realizes that this curve intersects the x-axis at $(1, 0)$. Choosing a few more values for t clarifies the shape of the curve even further, as shown below:

t	$x = 1 + 4t$	$y = 2t$
-2	$1 + 4(-2) = -7$	$2(-2) = -4$
-1	$1 + 4(-1) = -3$	$2(-1) = -2$
0	$1 + 4(0) = 1$	$2(0) = 0$
1	$1 + 4(1) = 5$	$2(1) = 2$
2	$1 + 4(2) = 9$	$2(2) = 4$



Interestingly, the constants in each variable function $x(t)$, and $y(t)$ of the parametrization correspond to the x-intercept, and the slope of the line ($m = 1/2$) is the same as the ratio between the derivatives of both variable functions ($[dy/dt]/[dx/dt]$). It is important to note that the constants in each variable function not always represent the x-intercept, but only a point in the curve where $t = 0$. These properties can be extended to all straight lines, so that the general parametric equation of the line can be written as follows:

$$x = x_0 + at, \; y = y_0 + bt, \text{ for } -\infty < t < \infty$$

As mentioned in the previous example the line passes through the point (x_0, y_0), and its slope corresponds to the ratio b/a. We will see later that it is not a coincidence that this ratio is determined by the derivatives of both x and y in terms of their parameter ($[dy/dt]/[dx/dt]$).

Example 2: Consider the following parametrization:

$$x = -1 + 3t, \; y = 1 - 6t, \text{ for } -\infty < t < \infty$$

Use the properties of the line parametrization to sketch its graph.

In some cases, the parameter can be removed to express the curve as a function. One way to find this function is solving for the parameter in one parametric equation to substitute it in the other. For example, in the previous problem, as shown below, we can solve for t using the equation related to x and then substitute the parameter in the equation for y,

$$\begin{aligned} -1 + 3t &= x \\ 3t &= x + 1 \\ t &= \frac{x+1}{3} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} y &= 1 - 6t \\ y &= 1 - 6\left(\frac{x+1}{3}\right) \\ y &= -2x - 1 \end{aligned}$$

The function that represents the curve is the slope-intercept form of the line, where one can confirm the value of the slope found in the previous problem.

The formula used for the slope of a straight line represented in parametric equations can also be extended to any parametric curve using the differentiation rules we previously studied. Using the chain rule conveniently we have that:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Solving for dy/dx, as long as dx/dt is not equal to zero gives us the following theorem.

Parametric Curves Derivative Theorem. *Suppose that a curve has the following parametrization:*

$$x = f(t),\ y = g(t),\ \textit{for } a \leqslant t \leqslant b$$

If both functions f and g are differentiable over the interval $[a, b]$, then the slope of the line tangent to the curve at the point $(f(t), g(t))$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0$$

It is important to note that in some cases instead of knowing the particular coordinates of a point, a problem might only provide the value of the parameter. An example of this problem is the following.

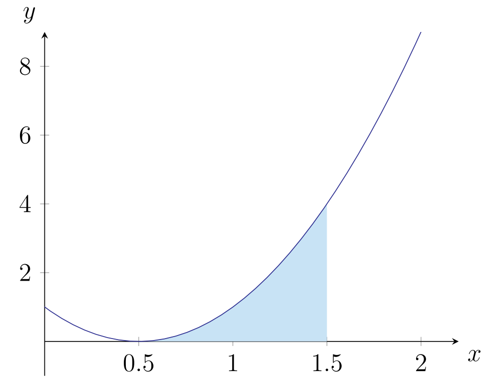
Example 3: Consider the following parametrization:

$$x = 1 - t,\ y = (1 - 2t)^2,\ \text{for } 0 \leqslant t \leqslant 2$$

a) Using the previous theorem, find the slope of the line tangent to this curve when $t = 1$

b) Rewrite the parametric curve in terms of one function and confirm the value found in part a)

As you might have guessed from the quadratic expression in y, the curve in the previous example is a parabola. The graph of this parabola is shown in the figure below.



Using the function found in part b) for a parametric curve, it is also possible to find the area under the graph of that curve using the definite integral. For example, the value of the area between $x = 0.5$ and $x = 1.5$ is equal to

$$\int_{0.5}^{1.5} y dx = \int_{0.5}^{1.5} (1 - 2(1 - x))^2 dx = \int_{0.5}^{1.5} (-1 + 2x)^2 dx = 4/3$$

Another version for calculating the integral for a parametric curve involves rewriting this integral in terms of t, using a generalization of the u-substitution previously studied, such substitution leads to the following result.

Area Under a Parametric Curve's Theorem. *Suppose that a curve over the x-axis has the following parametrization:*

$$x = f(t), \ y = g(t), \ \textit{for } a \leqslant t \leqslant b$$

.

If f is a differentiable function over the interval (a, b), then area under this curve is given by

$$A = \int_a^b g(t) f'(t)\, dt$$

Example 4: Compare the value previously found for the area under the parabola, with the value determine by using the parametric representation of the curve.