Computing Limits

MAC 2311

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In this section we will discuss techniques for computing limits for different types of functions and present a few basic rules to help us simplify the computation of limits.

1 Limit Laws

Assuming the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, then:

- **Sum**: $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- **Difference**: $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- **Constant Multiple**: $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x)$, where c is a constant
- **Product**: $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
- **Quotient**: $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (provided the limit of $\lim_{x\to a} g(x) \neq 0$)
- **Power**: $\lim_{x\to a}(f(x))^{\frac{n}{m}} = \left(\lim_{x\to a} f(x)\right)^{\frac{n}{m}}$, where n, m are integers with no common factors and $m \neq 0$

Note: These laws are also true for one-sided limits as $x \to a^-$ and $x \to a^+$.

Example 1: Given $\lim_{x\to 2} f(x) = 5$, $\lim_{x\to 2} g(x) = 8$, $\lim_{x\to 2} h(x) = 2$, compute $\lim_{x\to 2} \dfrac{f(x)^2}{g(x)-3h(x)}$. In your work, be sure to indicate each use of a limit law.

$\lim_{x\to 2} \dfrac{f(x)^2}{g(x)-3h(x)} =$

2 Common Limits

- The limit for a **constant** function, $f(x) = c$, for any constant c:

  $$\lim_{x \to a} f(x) = \lim_{x \to a} c = c$$

  regardless of the value of a

- The limit for the **identity** function, $f(x) = x$:

  $$\lim_{x \to a} f(x) = \lim_{x \to a} x = a$$

- The limit of a **polynomial** function, $p(x) = c_n x^n + \cdots + c_1 x + c_0$:

  $$\lim_{x \to a} p(x) = c_n a^n + \cdots + c_1 a + c_0 = p(a)$$

- The limit of a **rational** function (the quotient of two polynomials):

  $$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$$

  if $q(a) \neq 0$.

3 Techniques for Computing Limits

3.1 Direct Substitution

When presented with any limit, direct substitution is always the first step in trying to evaluate the limit. Direct substitution is simply plugging the x-value being approached directly into the function.

Example 2:

- $\lim_{x \to 2} (x^3 - x) = 2^3 - 2 = 8 - 2 = 6$
- $\lim_{x \to 0} e^x = e^0 = 1$
- $\lim_{x \to \frac{\pi}{2}} \cos(2x)$
- $\lim_{x \to 1} \frac{x}{x + 2}$
- $\lim_{x \to 3} 100$

3.2 Evaluating Limits of Functions as Quotients

Just as any other type of limit, when evaluating a limit that contains a quotient, trying direct substitution is the first step. If you substitute in the x-value that is being approached and the result is a real number, then you are done. However, in the event that the result of direct substitution yields an invalid output, there are some algebraic techniques we can use to determine the value of the limit, if it exists.

For rational functions, if after trying direct substitution you get a result of $\frac{0}{0}$ (an indeterminate form), then try factoring the numerator and/or denominator or try simplifying in order to find a common factor that can be canceled before trying direct substitution again.

Example 3: Evaluate the following: $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$

Let's try direct substitution first: $\frac{1^2+1-2}{1^2-1} \to \frac{0}{0}$

Since direct substitution results in the indeterminate form of $\frac{0}{0}$, let's try factoring the expression and canceling before utilizing direct substitution again.

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} &= \lim_{x\to 1} \frac{(x-1)(x+2)}{x(x-1)} \\
&= \lim_{x\to 1} \frac{x+2}{x} \\
&= \frac{1+2}{1} \\
&= 3
\end{aligned}$$

Example 4: Evaluate the following: $\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3}$

Direct substitution: $\frac{\frac{1}{3}-\frac{1}{3}}{3-3} \to \frac{0}{0}$ Simplify/factor expression:

$$\begin{aligned}
\lim_{x\to 3} \frac{\frac{1}{x}-\frac{1}{3}}{x-3} &= \lim_{x\to 3} \frac{\frac{3}{3x}-\frac{x}{3x}}{x-3} \\
&= \lim_{x\to 3} \frac{\frac{3-x}{3x}}{x-3} \\
&= \lim_{x\to 3} \frac{3-x}{3x(x-3)} \\
&= \lim_{x\to 3} \frac{-1(x-3)}{3x(x-3)} \\
&= \lim_{x\to 3} \frac{-1}{3x} \\
&= \frac{-1}{3\cdot 3} \\
&= -\frac{1}{9}
\end{aligned}$$

Example 5: Evaluate the following: $\lim_{x \to -3} \frac{x^2 + 4x + 3}{2x + 6}$

Example 6: Evaluate the following: $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$

Example 7: Evaluate the following: $\lim_{x \to 4} \frac{1 - \frac{4}{x}}{4 - x}$

For limits of quotient functions involving a square root in the numerator or denominator, if direct substitution leads to the indeterminate form of $\frac{0}{0}$, then we should try using the conjugate to rationalize the expression in order to find a common factor to cancel before utilizing direct substitution again.

Example 8: Evaluate the following: $\lim_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$

Direct substitution: $\dfrac{4-4}{\sqrt{4}-2} \to \dfrac{0}{0}$

Let's try using the conjugate to rationalize the expression, cancel a common factor, and then direct substitution.

$$\begin{aligned}
\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} &= \lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} \\
&= \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+2)}{x-4} \\
&= \lim_{x\to 4} (\sqrt{x}+2) \\
&= \sqrt{4}+2 \\
&= 4
\end{aligned}$$

Example 9: Evaluate the following: $\lim_{x\to 1} \dfrac{\sqrt{x+8}-3}{x-1}$