MAP 2302 WRITTEN HOMEWORK #3

Question 1. Find an explicit general solution of the differential equation.

$$(x^4 - x + y)\,dx + (-x)dy = 0$$

First check if DE is exact by checking if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$. So DE is <u>not</u> exact, but $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ has a nice relation to N, specifically $\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)/N = \frac{2}{-x}$ depends only on x. So look for integrating factor $\mu(x)$.

Want $\mu M\,dx + \mu N\,dy$ to be exact

Want $\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$

μ is independent of y

$$\mu \cdot \frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x} \cdot N + \mu \cdot \frac{\partial N}{\partial x}$$

$$\mu \cdot 1 = \frac{\partial \mu}{\partial x} \cdot (-x) + \mu \cdot (-1)$$

$$2\mu = -\frac{\partial \mu}{\partial x} \cdot x \Rightarrow 2\mu = -\frac{d\mu}{dx} \cdot x$$

Switched from ∂ to d
because μ is a function of <u>one</u> variable

$$\frac{2}{x}\,dx = -\frac{d\mu}{\mu}$$

$$\int \frac{2}{x}\,dx = -\int \frac{d\mu}{\mu}$$

$$2\ln x = -\ln \mu$$

$$-2\ln x = \ln \mu$$

$$e^{-2\ln x} = e^{\ln \mu}$$

$$\left(e^{\ln x}\right)^{-2} = e^{\ln \mu}$$

$$x^{-2} = \mu$$

So an integrating factor is $\mu = x^{-2}$. Multiply both sides of the original DE by x^{-2}.

$$x^{-2}(x^4 - x + y)\,dx + x^{-2}(-x)\,dy = 0$$ This will be exact

$$\underbrace{(x^2 - x^{-1} + x^{-2}y)}_{\text{New } M}\,dx + \underbrace{(-x^{-1})}_{\text{New } N}\,dy = 0$$

Take antideriv. wrt x

Take antiderivative with respect to y

$$F = \frac{x^3}{3} - \ln|x| + \frac{x^{-1}}{-1}y + \underbrace{A(y)}_{\text{This can be } 0}$$

$$F = -x^{-1}y + \underbrace{B(x)}_{\text{This can be } \frac{x^3}{3} - \ln|x|}$$

$$F = \frac{x^3}{3} - \ln|x| - \frac{y}{x}$$ works.

So an implicit general solution to the DE is

$$\frac{x^3}{3} - \ln|x| - \frac{y}{x} = C$$

which we can rearrange into an <u>explicit</u> solution as follows.

$$\frac{x^3}{3} - \ln|x| \underbrace{- C}_{\text{can rename}} = \frac{y}{x}$$

$$\boxed{\frac{x^4}{3} - x\ln|x| + Cx = y}$$

Question 2. Find an explicit general solution of the differential equation.

$$\frac{dy}{dx} = \frac{2y}{x} - x^2y^2 \quad \Rightarrow \quad \frac{dy}{dx} - \frac{2y}{x} = -x^2y^2$$

This DE has the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$ (Bernoulli)

where $P(x) = -\frac{2}{x}$, $Q(x) = -x^2$, and $n = 2$.

Start by multiplying both sides by $y^{-n} = y^{-2}$. $v = y^{1-n} = y^{-n+1}$

$$y^{-2}\frac{dy}{dx} - \frac{2}{x}y^{-1} = -x^2$$

Next, substitute $v = y^{-1}$

$$\downarrow$$

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$$

$$-\frac{dv}{dx} - \frac{2}{x}v = -x^2$$

$$\frac{dv}{dx} + \frac{2}{x}v = x^2$$

This is a linear differential equation. We can find an integrating factor.

$P(x) = \frac{2}{x}$

$\int P(x)dx = 2\ln x$

$e^{\int P(x)dx} = e^{2\ln x} = x^2$

Multiply by x^2

$$x^2\frac{dv}{dx} + 2xv = x^4$$

$$\frac{d}{dx}\left(x^2v\right) = x^4$$

$$x^2 v = \int x^4\,dx$$

$$x^2 v = \frac{x^5}{5} + C$$

$$5x^2 v = x^5 + \underbrace{5C}_{\text{can rename}}$$

$$5x^2 v = x^5 + C$$

$$v = \frac{x^5 + C}{5x^2}$$

$$y^{-1} = \frac{x^5 + C}{5x^2}$$

$$\boxed{y = \frac{5x^2}{x^5 + C}}$$