MAP 2302 WRITTEN HOMEWORK #7

Question 1. Find a particular solution of the differential equation.

$$y'' - 2y' + y = t^{-1}e^t$$

Associated homog DE is $y'' - 2y' + y = 0 \Rightarrow$ Aux eqn $r^2 - 2r + 1 = 0$
$\Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$ is a double root.

Two independent solns of the homog DE are $y_1 = e^t$ and $y_2 = te^t$.

That is, we have $y_1'' - 2y_1' + y_1 = 0$ and $y_2'' - 2y_2' + y_2 = 0$.

We'll look for a solution of the nonhomog DE of the form

$$y = v_1 y_1 + v_2 y_2 \quad \text{where } v_1, v_2 \text{ are functions}$$

$$\downarrow$$

$$y' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

Impose the condition

$$v_1' y_1 + v_2' y_2 = 0 \quad \text{(i)}$$

$$y' = v_1 y_1' + v_2 y_2'$$

$$\downarrow$$

$$y'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Next, substitute y'', y', and y into the nonhomog DE

$$\overbrace{v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''}^{y''} - 2\overbrace{(v_1y_1' + v_2y_2')}^{y'} + \overbrace{v_1y_1 + v_2y_2}^{y} = t^{-1}e^t$$

Now use the fact that $y_1'' - 2y_1' + y_1 = 0 \Rightarrow v_1y_1'' - 2v_1y_1' + v_1y_1 = 0$

$$y_2'' - 2y_2' + y_2 = 0 \Rightarrow v_2y_2'' - 2v_2y_2' + v_2y_2 = 0$$

$$v_1'y_1' + v_2'y_2' = t^{-1}e^t \quad \text{(ii)}$$

$$y_1 = e^t \Rightarrow y_1' = e^t \qquad y_2 = te^t \Rightarrow y_2' = e^t + te^t$$

$$\text{(i)} \Rightarrow e^t v_1' + te^t v_2' = 0$$

$$\text{(ii)} \Rightarrow e^t v_1' + (e^t + te^t) v_2' = t^{-1}e^t$$

$$\text{(ii)} - \text{(i)} \Rightarrow e^t v_2' = t^{-1}e^t \Rightarrow v_2' = t^{-1} \Rightarrow v_2 = \ln t$$

Sub $v_2' = t^{-1}$ into (i): $v_1' = -1 \Rightarrow v_1 = -t$

CONCLUSION: A solution to the nonhomog DE is

$$y = v_1y_1 + v_2y_2 = -te^t + \ln t \cdot te^t$$

We could DOUBLE-CHECK this using Wolfram Alpha!

Question 2. Verify that $y = t^{-2}$ is a solution of the differential equation.

$$t^2y'' + 5ty' + 4y = 0$$

Then, find another (independent) solution of the form $y = t^{-2}v(t)$. (Hint: Substitute $y = t^{-2}v(t)$ and get a separable differential equation in $w = v'$.)

(i) Verify. $y = t^{-2} \Rightarrow y' = -2t^{-3} \Rightarrow y'' = 6t^{-4}$

$$\text{LHS} = t^2 \cdot 6t^{-4} + 5t\cdot(-2)t^{-3} + 4\cdot t^{-2}$$

$$= 6t^{-2} - 10t^{-2} + 4t^{-2} = 0 \quad \checkmark$$

(ii) Look for a solution of the form $y = t^{-2}v$

$$\Rightarrow y' = -2t^{-3}v + t^{-2}v'$$

$$\Rightarrow y'' = 6t^{-4}v - 2t^{-3}v' - 2t^{-3}v' + t^{-2}v''$$

$$= 6t^{-4}v - 4t^{-3}v' + t^{-2}v''$$

Plug y'', y', and y into the DE:

$$t^2\cdot(6t^{-4}v - 4t^{-3}v' + t^{-2}v'') + 5t\cdot(-2t^{-3}v + t^{-2}v') + 4(t^{-2}v) = 0$$

$$6t^{-2}v - 4t^{-1}v' + v'' - 10t^{-2}v + 5t^{-1}v' + 4t^{-2}v = 0$$

$$\Rightarrow v'' + t^{-1}v' = 0 \qquad \text{Let } w = v'$$

$$w' + t^{-1}w = 0 \quad \Rightarrow \quad \frac{dw}{dt} + t^{-1}w = 0$$

$$\Rightarrow \quad \frac{dw}{dt} = -\frac{1}{t} w$$

$$\frac{1}{w} dw = -\frac{1}{t} dt$$

$$\int \frac{1}{w} dw = -\int \frac{1}{t} dt$$

$$\ln w = -\ln t$$

$$e^{\ln w} = e^{-\ln t} = 1/e^{\ln t}$$

$$w = 1/t = t^{-1} \quad \Rightarrow v' = \frac{1}{t}$$

$$\Rightarrow v = \ln t$$

CONCLUSION: $y = t^{-2} v = t^{-2} \ln t$ is a solution.

We can DOUBLE-CHECK this!

$$y = t^{-2} \ln t$$

$$\downarrow$$

$$y' = -2t^{-3} \ln t + t^{-2} \cdot t^{-1} = -2t^{-3} \ln t + t^{-3}$$

$$\downarrow$$

$$y'' = 6t^{-4} \ln t - 2t^{-3} \cdot t^{-1} - 3t^{-4} = 6t^{-4} \ln t - 5t^{-4}$$

$$\Rightarrow \left.\begin{aligned} t^2 y'' &= 6t^{-2} \ln t - 5t^{-2} \\ 5t y' &= -10t^{-2} \ln t + 5t^{-2} \\ 4y &= 4t^{-2} \ln t \end{aligned}\right\} \text{these add to 0, as they should}$$