MAP 2302 WRITTEN HOMEWORK #8

Question 1. Find the first three nonzero terms in the Taylor series for the solution to the initial value problem.

$$y' = y^2, \qquad y(0) = 1$$

$$y'' = 2y \cdot y'$$

So $y(0) = 1$

$y'(0) = y(0)^2 = 1^2 = 1$

$y''(0) = 2y(0)y'(0) = 2 \cdot 1 \cdot 1 = 2$

First three nonzero terms are

$$y(0) + y'(0)x + \frac{y''(0)}{2!}x^2$$

$$= 1 + 1x + \frac{2}{2!}x^2$$

$$= 1 + x + x^2$$

More on question 1

We got $1+x+x^2$ for the first three terms. Does this pattern continue?

Can we solve the D.E. another way?

$\frac{dy}{dx} = y^2$ separable $\Rightarrow$ $y^{-2}dy = dx$

$$\int y^{-2}dy = \int dx$$

$$\frac{y^{-1}}{-1} = x + C$$

$$y^{-1} = -x - C = -x + C \text{ (rename)}$$

$$y = \frac{1}{-x+C}$$

Initial condition says if $x=0$ then $y=1$. This implies $C=1$.

Solution to initial value problem is $y = \frac{1}{1-x}$

As an infinite series, this is $1+x+x^2+x^3+\ldots$

(geometric series)

Converges for only some values of x, namely $-1 < x < 1$

Question 2. Consider the following initial value problem.

$$y' = x + y, \qquad y(0) = 0$$

(a) Find the solution in the form of an infinite series.

(b) Find an exact formula for the solution using techniques from earlier in the course.

(a)
$$\begin{aligned}
y' &= x + y & &\Rightarrow y'(0) = 0 + y(0) = 0 + 0 = 0 \\
y'' &= 1 + y' & & y''(0) = 1 + y'(0) = 1 + 0 = 1 \\
y''' &= y'' & & y'''(0) = y''(0) = 1 \\
y^{(4)} &= y''' & & y^{(4)}(0) = y'''(0) = 1 \\
&\text{etc.} & & \text{etc.}
\end{aligned}$$

So the solution is $y(0) + y'(0) \cdot x + \frac{y''(0)}{2!} \cdot x^2 + \frac{y'''(0)}{3!} \cdot x^3 + \dots$

$$= 0 + 0 \cdot x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \dots$$

or $\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$ or $\sum_{n=2}^{\infty} \frac{x^n}{n!}$

(b) $y' - y = x$ Nonhomogeneous, linear, constant coefficients

Auxiliary equation $r - 1 = 0 \Rightarrow r = 1$

A solution to the homogeneous DE ($y' - y = 0$) is $y = Ce^{1x} = Ce^{x}$

To find a particular solution of the nonhomog DE, try $y = Ax + B$
$$\downarrow$$
$$y' = A$$

$$A - (Ax + B) = x$$

$$(-A)x + (A - B) = 1x + 0 \Rightarrow \begin{matrix} -A = 1 \\ A - B = 0 \end{matrix} \Rightarrow \begin{matrix} A = -1, \\ B = -1 \end{matrix}$$

So a particular solution to the nonhomog DE is $y = -x - 1$

The general solution to the nonhomog DE is $y = Ce^{x} - x - 1$

Next, use initial condition to find C.

$$y(0) = 0 \Rightarrow Ce^{0} - 0 - 1 = 0 \Rightarrow C - 1 = 0 \Rightarrow C = 1$$

The solution to the initial value problem is $y = e^{x} - x - 1$.

Note that this is consistent with our answer to part (a).

We know $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

So $e^{x} - x - 1 = \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$