MAP 2302 WRITTEN HOMEWORK #9

Question 1. Find the first three nonzero terms in the Taylor series for the solution to the initial value problem.

$$y'' = -\cos y, \qquad y(0) = 0, \qquad y'(0) = 1$$

$$\downarrow$$

$$y''' = \sin y \cdot y' \quad \text{(CHAIN RULE)}$$

$$\downarrow$$

$$y^{(4)} = (\sin y)' \cdot y' + \sin y \cdot (y')' \quad \text{(PRODUCT RULE)}$$

$$= \cos y \cdot y' \cdot y' + \sin y \cdot y''$$

$$y''(0) = -\cos y(0) = -\cos 0 = -1$$

$$y'''(0) = \sin y(0) \cdot y'(0) = \sin 0 \cdot 1 = 0$$

$$y^{(4)}(0) = \underbrace{\cos y(0)}_{\substack{\cos 0 \\ =1}} \cdot \underbrace{y'(0)}_{1} \cdot \underbrace{y'(0)}_{1} + \underbrace{\sin y(0)}_{\substack{\sin 0 \\ =0}} \cdot y''(0) = 1$$

$$y = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \dots$$

$$= 0 + \underbrace{\frac{1}{1!}x}_{\text{nonzero}} + \underbrace{\frac{-1}{2!}x^2}_{\text{nonzero}} + \frac{0}{3!}x^3 + \underbrace{\frac{1}{4!}x^4}_{\text{nonzero}} + \dots$$

$$\text{or} \quad x - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{or} \quad x - \frac{x^2}{2} + \frac{x^4}{24}$$

Question 2. Find the first six nonzero terms in the Taylor series for a general solution to the differential equation.

$$y'' + xy' + y = 0 \quad \text{Second order}$$

METHOD 1. No initial values given, so say $y(0) = c_1$ and $y'(0) = c_2$

$$\left.\begin{aligned} y'' &= -xy' - y \\ &\downarrow \\ y''' &= -y' - xy'' - y' \\ &= -xy'' - 2y' \\ y^{(4)} &= -y'' - xy''' - 2y'' \\ &= -xy''' - 3y'' \\ y^{(5)} &= -y''' - xy^{(4)} - 3y''' \\ &= -xy^{(4)} - 4y''' \end{aligned}\right\} \Rightarrow \begin{aligned} & y''(0) = -0y'(0) - y(0) \Rightarrow y''(0) = -y(0) = -c_1 \\ & y'''(0) = -0y''(0) - 2y'(0) \Rightarrow y'''(0) = -2y'(0) = -2c_2 \\ & y^{(4)}(0) = -0y'''(0) - 3y''(0) \Rightarrow y^{(4)}(0) = -3y''(0) = -3(-c_1) = 3c_1 \\ & y^{(5)}(0) = -0y^{(4)}(0) - 4y'''(0) \Rightarrow y^{(5)}(0) = -4y'''(0) = -4\cdot(-2c_2) = 8c_2 \end{aligned}$$

The first six nonzero terms are

$$y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5$$

$$= c_1 + \frac{c_2}{1}x + \frac{-c_1}{2}x^2 + \frac{-2c_2}{6}x^3 + \frac{3c_1}{24}x^4 + \frac{8c_2}{120}x^5$$

or $c_1 + c_2x - \frac{c_1}{2}x^2 - \frac{c_2}{3}x^3 + \frac{c_1}{8}x^4 + \frac{c_2}{15}x^5$

or $c_1\cdot\left(1 - \frac{x^2}{2} + \frac{x^4}{8}\right) + c_2\cdot\left(x - \frac{x^3}{3} + \frac{x^5}{15}\right)$

Question 2. Find the first six nonzero terms in the Taylor series for a general solution to the differential equation.

$$y'' + xy' + y = 0$$

METHOD 2. Say $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots$

$\downarrow$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \dots$

$\downarrow$

$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots$

$$\Rightarrow \begin{aligned} y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \dots \\ xy' &= a_1 x + 2a_2 x^2 + 3a_3 x^3 + \dots \\ y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \end{aligned}$$

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$$0 = \underbrace{(2a_2 + a_0)}_{0} + \underbrace{(6a_3 + 2a_1)}_{0} x + \underbrace{(12a_4 + 3a_2)}_{0} x^2 + \underbrace{(20a_5 + 4a_3)}_{0} x^3 + \dots$$

$a_0 = a_0$ and $a_1 = a_1$. $2a_2 + a_0 = 0 \Rightarrow 2a_2 = -a_0 \Rightarrow a_2 = -\frac{1}{2}a_0$

$6a_3 + 2a_1 = 0 \Rightarrow 6a_3 = -2a_1 \Rightarrow a_3 = -\frac{1}{3}a_1$

$12a_4 + 3a_2 = 0 \Rightarrow 12a_4 = -3a_2 \Rightarrow a_4 = -\frac{1}{4}a_2 = -\frac{1}{4}\cdot\left(-\frac{1}{2}a_0\right) = \frac{1}{8}a_0$

$20a_5 + 4a_3 = 0 \Rightarrow 20a_5 = -4a_3 \Rightarrow a_5 = -\frac{1}{5}a_3 = -\frac{1}{5}\cdot\left(-\frac{1}{3}a_1\right) = \frac{1}{15}a_1$

The first six nonzero terms are

$$a_0 + a_1 x - \frac{a_0}{2}x^2 - \frac{a_1}{3}x^3 + \frac{a_0}{8}x^4 + \frac{a_1}{15}x^5$$