MAP 2302 WRITTEN HOMEWORK #12

Question 1. Find the inverse Laplace transform of $\frac{7s^2+23s+30}{(s-2)(s^2+2s+5)}$.

$$F(s) = \frac{7s^2+23s+30}{(s-2)(s^2+2s+5)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+2s+5}$$

$$\downarrow$$

$$7s^2+23s+30 = A(s^2+2s+5) + (Bs+C)(s-2)$$

Plug in $s=2$ to find A:

$$\underbrace{7\cdot 4}_{28} + \underbrace{23\cdot 2}_{46} + 30 = A(4+4+5) + 0$$

$$104 = 13A \Rightarrow A = 8.$$ Next, find B and C

So now we have

$$7s^2+23s+30 = 8(s^2+2s+5)+(Bs+C)(s-2)$$
$$= 8s^2+16s+40+Bs^2-2Bs+Cs-2C$$
$$= (8+B)s^2+(16-2B+C)s+(40-2C)$$

$$\Rightarrow \left.\begin{aligned} 7 &= 8+B \\ 23 &= 16-2B+C \\ 30 &= 40-2C \end{aligned}\right\} \Rightarrow B=-1,\ C=5$$

$$\Rightarrow F(s) = \frac{8}{s-2} + \frac{-s+5}{s^2+2s+5}$$

$$= \frac{8}{s-2} + \frac{-s+5}{(s+1)^2+2^2} = \frac{8}{s-2} + \frac{-s-1}{(s+1)^2+2^2} + \frac{6}{(s+1)^2+2^2}$$

$$= 8\cdot\frac{1}{s-2} - \frac{s+1}{(s+1)^2+2^2} + 3\cdot\frac{2}{(s+1)^2+2^2}$$

$$\Rightarrow f(t) = 8e^{2t} - e^{-t}\cos 2t + 3e^{-t}\sin 2t$$

Question 2. Find $\mathcal{L}^{-1}\{F\}$, given $s^2F(s) - 4F(s) = \dfrac{5}{s+1}$.

$$(s^2-4)F(s) = \frac{5}{s+1}$$

$$F(s) = \frac{5}{(s+1)(s^2-4)} = \frac{5}{(s+1)(s+2)(s-2)}$$

Partial fractions: $\dfrac{5}{(s+1)(s+2)(s-2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2} + \dfrac{C}{s-2}$

$$\Rightarrow 5 = A(s+2)(s-2) + B(s+1)(s-2) + C(s+1)(s+2)$$

$s=-1$: $5 = A\cdot 1\cdot(-3) + 0 + 0$ $\qquad 5 = -3A$

$s=-2$: $5 = 0 + B\cdot(-1)\cdot(-4) + 0$ $\qquad 5 = 4B$

$s=2$: $5 = 0 + 0 + C\cdot 3\cdot 4$ $\qquad 5 = 12C$

$$A = -\frac{5}{3}, \quad B = \frac{5}{4}, \quad C = \frac{5}{12}$$

$$F(s) = -\frac{5}{3}\cdot\frac{1}{s+1} + \frac{5}{4}\cdot\frac{1}{s+2} + \frac{5}{12}\cdot\frac{1}{s-2}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{5}{3}e^{-t} + \frac{5}{4}e^{-2t} + \frac{5}{12}e^{2t}$$

Question 3. Solve the initial value problem using Laplace transforms.

$$y'' + 6y' + 9y = 0, \qquad y(0) = -1, \qquad y'(0) = 6$$

$$\mathcal{L}\{y'' + 6y' + 9y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} + 6\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = 0$$

$$\underbrace{s^2F(s) - s\cdot\overset{-1}{\widetilde{y(0)}} - \overset{6}{\widetilde{y'(0)}}}_{\mathcal{L}\{y''\}} + 6\big(\underbrace{sF(s) - \overset{-1}{\widetilde{y(0)}}}_{\mathcal{L}\{y'\}}\big) + \underbrace{9F(s)}_{\mathcal{L}\{y\}} = 0$$

$$s^2F(s) + s - 6 + 6sF(s) + 6 + 9F(s) = 0$$

$$s^2F(s) + 6sF(s) + 9F(s) = -s$$

$$(s^2 + 6s + 9)F(s) = -s$$

$$F(s) = \frac{-s}{s^2+6s+9} = \frac{-s}{(s+3)^2} = \frac{A}{s+3} + \frac{B}{(s+3)^2}$$

$\downarrow$ multiply by $(s+3)^2$

$$-s = A(s+3) + B \Rightarrow \begin{array}{l} A = -1 \\ B = 3 \end{array}$$

$$F(s) = \frac{-1}{s+3} + \frac{3}{(s+3)^2}$$

$$y = \mathcal{L}^{-1}\left\{\frac{-1}{s+3} + \frac{3}{(s+3)^2}\right\} = -e^{-3t} + 3te^{-3t}$$