WRITE YOUR NAME:

MAP 2302 Test 1 Tuesday September 24th
Total possible score: 18 points

Question 1. Verify that the function $y = xe^{7x}$ is a solution of the differential equation $y' - 7y = e^{7x}$.

$$y = x \cdot e^{7x} \Rightarrow y' = (x)' e^{7x} + x(e^{7x})'$$

$$= 1e^{7x} + x \cdot e^{7x} \cdot 7$$

$$= e^{7x} + 7xe^{7x}$$

$$LHS = y' - 7y = (\underbrace{e^{7x} + 7xe^{7x}}_{y'}) - 7(\underbrace{xe^{7x}}_{y})$$

$$= e^{7x} + \underbrace{7xe^{7x} - 7xe^{7x}}_{=0}$$

$$= e^{7x} = RHS \checkmark$$

Question 2. Find an explicit general solution of the differential equation.

$$\frac{dy}{dx} = 5x^4y$$

Separable

$$\frac{1}{y}\,dy = 5x^4\,dx$$

$$\int \frac{1}{y}\,dy = \int 5x^4\,dx$$

$$\ln|y| = x^5 + C$$

$$|y| = e^{x^5+C} = e^{x^5} \cdot \underbrace{e^{C}}_{\text{can rename if you want}}$$

$$y = \underbrace{\pm e^{C}}_{\text{can rename}} \cdot e^{x^5}$$

$$\boxed{y = Ce^{x^5}}$$

Question 3. Find an explicit general solution of the differential equation.

$$y' + \frac{1}{x}y = 3x + 8$$ Linear

$$y' + \underbrace{\left(\frac{1}{x}\right)}_{P(x)} y = \underbrace{(3x+8)}_{Q(x)}$$

We know we can use $e^{\int P(x)dx}$ as an integrating factor (just need one)

$$P(x) = \frac{1}{x}$$

$$\int P(x)dx = \ln x$$

$$\mu(x) = e^{\int P(x)dx} = e^{\ln x} = x$$ ← integrating factor. Multiply both sides of DE by x.

$$\Rightarrow \underset{\substack{\uparrow \\ \mu = x}}{x} y' + \underset{\substack{\uparrow \\ \mu' = 1}}{} y = 3x^2 + 8x$$

$$\frac{d}{dx}(xy) = 3x^2 + 8x$$

$$xy = \int (3x^2 + 8x)\,dx$$

$$xy = x^3 + 4x^2 + C$$

$$\boxed{y = x^2 + 4x + \frac{C}{x}}$$

Question 4. Find an explicit solution of the initial value problem.

$$\frac{dy}{dx} = 3x^2(y-2), \qquad y(0) = 7$$

Separable

$$\frac{1}{y-2}\,dy = 3x^2\,dx$$

$$\int \frac{1}{y-2}\,dy = \int 3x^2\,dx$$

$$\ln|y-2| = x^3 + C$$

$$|y-2| = e^{x^3+C} = e^{x^3}\cdot \underbrace{e^C}_{\text{could rename}}$$

$$y-2 = \underbrace{\pm e^C}_{\text{rename}}\cdot e^{x^3} = C\cdot e^{x^3}$$

$$y = C\cdot e^{x^3} + 2$$

Initial condition says if $x=0$ then $y=7$

$$7 = C\cdot \underbrace{e^0}_{1} + 2 \quad \Rightarrow C=5$$

ANSWER: $y = 5e^{x^3} + 2$

Question 5. Find an explicit solution of the initial value problem.

$$\underbrace{(2xy+5)}_{M}\,dx + \underbrace{(x^2-1)}_{N}\,dy = 0, \qquad y(2) = 3$$

Is this exact? $\dfrac{\partial M}{\partial y} = 2x \quad \dfrac{\partial N}{\partial x} = 2x$ Yes!

$M = 2xy + 5$ $\qquad\qquad$ $N = x^2 - 1$

$\downarrow$ integrate wrt x $\qquad\qquad$ $\downarrow$ integrate wrt y

$F = x^2y + 5x + g(y)$ $\qquad\qquad$ $F = x^2y - y + h(x)$

Can use $F = x^2y + 5x - y$.

General solution is $x^2y + 5x - y = C$.

Find C using initial condition. If $x=2$ then $y=3$.

$$2^2 \cdot 3 + 5 \cdot 2 - 3 = C$$
$$12 + 10 - 3 = C \Rightarrow C = 19$$

Solution is $x^2y + 5x - y = 19$

$$x^2y - y = 19 - 5x$$
$$(x^2-1)y = 19 - 5x$$
$$y = \frac{19-5x}{x^2-1} \quad \text{or} \quad \frac{5x-19}{1-x^2}$$

Question 6. Solve the differential equation.

$$(2xy)\,dx + (y^2 - 3x^2)\,dy = 0$$

HINT: Start by multiplying both sides by y^{-4}.

$$\underbrace{(2xy^{-3})}_{M}\,dx + \underbrace{(y^{-2} - 3x^2y^{-4})}_{N}\,dy = 0$$

Is it exact? $\frac{\partial M}{\partial y} = -6xy^{-4}$ $\quad \frac{\partial N}{\partial x} = -6xy^{-4}$ Yes!

$M = 2xy^{-3}$

$\downarrow$ integrate wrt x

$F = x^2y^{-3} + A(y)$

$N = y^{-2} - 3x^2y^{-4}$

$\downarrow$ integrate wrt y

$$F = \frac{y^{-1}}{-1} - 3x^2\frac{y^{-3}}{-3} + B(x)$$

$$= -y^{-1} + x^2y^{-3} + B(x)$$

Can use $F = x^2y^{-3} - y^{-1}$

Solution is $x^2y^{-3} - y^{-1} = C$ or $\frac{x^2}{y^3} - \frac{1}{y} = C$

Question 7. Solve the initial value problem.

$$y'' - 4y' - 5y = 0, \qquad y(0) = 6, \qquad y'(0) = 6$$

Auxiliary eqn: $r^2 - 4r - 5 = 0$

$(r+1)(r-5) = 0 \qquad r = 5, r = -1$

General solution: $y = Ae^{5t} + Be^{-t}$

$y' = 5Ae^{5t} - Be^{-t}$

$y(0) = Ae^0 + Be^0 = A + B$

$y'(0) = 5Ae^0 - Be^0 = 5A - B$

$$\begin{array}{rl} A + B &= 6 \\ 5A - B &= 6 \\ \hline 6A &= 12 \end{array} \Rightarrow A = 2$$

$\downarrow$

$B = 4$

Solution: $y = 2e^{5t} + 4e^{-t}$

Question 8. Solve the initial value problem.

$$y'' - 6y' + 9y = 0, \qquad y(0) = 2, \qquad y'(0) = 7$$

Auxiliary equation: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$ $\quad r = 3$ is a double root

General solution: $y(t) = Ae^{3t} + Bte^{3t}$

$\downarrow$

$y'(t) = 3Ae^{3t} + Be^{3t} + 3Bte^{3t}$

$y(0) = A \cdot 1 + 0 = A$ $\qquad A = 2$

$y'(0) = 3A \cdot 1 + B \cdot 1 + 0 = 3A + B$ $\qquad 3A + B = 7$

$\Rightarrow A = 2,\ B = 1$

Solution: $y = 2e^{3t} + te^{3t}$

Question 9. Solve the initial value problem.

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}, \qquad y(1) = \frac{1}{2}$$

HINT: Start with the substitution $v = y/x$.

$$\frac{dy}{dx} = \frac{xy}{x^2} - \frac{y^2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

$$v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v'x + vx' = x\frac{dv}{dx} + v$$

DE becomes $x\frac{dv}{dx} + v = v - v^2 \Rightarrow x\frac{dv}{dx} = -v^2$

$$\Rightarrow v^{-2}\,dv = -\frac{1}{x}\,dx \Rightarrow \int v^{-2}\,dv = \int -\frac{1}{x}\,dx$$

$$\Rightarrow \frac{v^{-1}}{-1} = -\ln|x| + C \Rightarrow v^{-1} = \ln|x| + C$$

renamed

$$\Rightarrow v = \frac{1}{\ln|x| + C} \Rightarrow \frac{y}{x} = \frac{1}{\ln|x| + C}$$

$$\Rightarrow y = \frac{x}{\ln|x| + C}$$

If $x = 1$ then $y = \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{\underbrace{\ln 1}_{0} + C} \Rightarrow C = 2$$

SOLUTION: $$y = \frac{x}{\ln|x| + 2}$$