WRITE YOUR NAME:

MAP 2302 Test 2 Tuesday October 22nd
Total possible score: 18 points

Question 1. Find the **general** solution of the differential equation.

$$y'' + 6y' + 10y = 0$$

Linear homogeneous with constant coefficients.

Auxiliary equation: $r^2 + 6r + 10 = 0$

$$r^2 + 6r + 9 = -1$$

$$(r+3)^2 = -1$$

$$r+3 = \pm\sqrt{-1} = \pm i$$

$$r = -3 \pm i \qquad \alpha \pm \beta i$$

Two independent real-valued solutions are $\alpha = -3,\ \beta = 1$

$$y = e^{-3t}\cos t\,,\quad y = e^{-3t}\sin t \qquad \{e^{\alpha t}\cos\beta t,\ e^{\alpha t}\sin\beta t\}$$

General solution:

$$y = c_1 e^{-3t}\cos t + c_2 e^{-3t}\sin t$$

$$\text{or } e^{-3t}(c_1\cos t + c_2 \sin t)$$

Question 2. Find a **particular** solution of the differential equation.

$$y'' - y' + 25y = 5\sin 5t$$

Try a solution of the form $y = A\sin 5t + B\cos 5t$

$$\downarrow$$

$$y' = 5A\cos 5t - 5B\sin 5t$$

$$\downarrow$$

$$y'' = -25A\sin 5t - 25B\cos 5t$$

Plug these into the DE:

$$\underbrace{(-25A\sin 5t - 25B\cos 5t)}_{y''} - \underbrace{(5A\cos 5t - 5B\sin 5t)}_{y'} + 25\underbrace{(A\sin 5t + B\cos 5t)}_{y} = 5\sin 5t$$

$$-25A\sin 5t - 25B\cos 5t - 5A\cos 5t + 5B\sin 5t + 25A\sin 5t + 25B\cos 5t = 5\sin 5t$$

$$(-25A + 5B + 25A)\sin 5t + (-25B - 5A + 25B)\cos 5t = 5\sin 5t$$

$$5B\sin 5t \quad - 5A\cos 5t \quad = \quad 5\sin 5t$$

$$\Rightarrow 5B = 5, \quad -5A = 0 \qquad \Rightarrow B = 1, \quad A = 0$$

A particular solution is $y = \cos 5t$

Question 3. Solve the initial value problem.

$$y'' - 2y' + 5y = 0, \qquad y(0) = 1, \qquad y'(0) = 17$$

Auxiliary equation $r^2 - 2r + 5 = 0 \Rightarrow r^2 - 2r + 1 = -4$

$\Rightarrow (r-1)^2 = -4 \Rightarrow r - 1 = \pm\sqrt{-4} = \pm 2i \Rightarrow r = 1 \pm 2i$

$\alpha \pm \beta i$ $\quad \alpha = 1$, $\beta = 2$

General solution $y = c_1 e^t \sin 2t + c_2 e^t \cos 2t$

PRODUCT PRODUCT

$\Rightarrow$

$y' = c_1 e^t \sin 2t + 2c_1 e^t \cos 2t + c_2 e^t \cos 2t - 2c_2 e^t \sin 2t$

$1 = y(0) = c_1 e^0 \underbrace{\sin 0}_{0} + c_2 \underbrace{e^0}_{1} \underbrace{\cos 0}_{1} \Rightarrow c_2 = 1$

$17 = y'(0) = c_1 e^0 \underbrace{\sin 0}_{0} + 2c_1 \underbrace{e^0}_{1} \underbrace{\cos 0}_{1} + c_2 \underbrace{e^0}_{1} \underbrace{\cos 0}_{1} - 2c_2 e^0 \underbrace{\sin 0}_{0}$

$\Rightarrow 2c_1 + c_2 = 17 \Rightarrow 2c_1 = 16 \Rightarrow c_1 = 8$

Answer: $y = 8e^t \sin 2t + e^t \cos 2t$

or $e^t(8 \sin 2t + \cos 2t)$

Question 4. Find the **general** solution of the differential equation.

$$y'' - 7y' + 12y = 10e^{2t}$$

(i) Associated homogeneous equation is $y'' - 7y' + 12y = 0$

Auxiliary equation is $r^2 - 7r + 12 = 0 \Rightarrow (r-3)(r-4) = 0$

$\Rightarrow r = 3, r = 4 \Rightarrow$ general solution of homog. DE is $c_1 e^{3t} + c_2 e^{4t}$

(ii) Find a particular solution of the nonhomogeneous equation.

Try $y = Ae^{2t}$

$\downarrow$

$y' = 2Ae^{2t}$

$\downarrow$

$y'' = 4Ae^{2t}$

Plug these into the nonhomog DE:

$$\underbrace{4Ae^{2t}}_{y''} - 7 \cdot \underbrace{2Ae^{2t}}_{y'} + 12 \cdot \underbrace{Ae^{2t}}_{y} = 10e^{2t}$$

$$4Ae^{2t} - 14Ae^{2t} + 12Ae^{2t} = 10e^{2t}$$

$$(4 - 14 + 12)Ae^{2t} = 10e^{2t}$$

$$2Ae^{2t} = 10e^{2t}$$

$$2A = 10$$

$$A = 5$$

A particular solution of the nonhomog DE is $y = 5e^{2t}$

The general solution of the nonhomog DE is:

$$y = c_1 e^{3t} + c_2 e^{4t} + 5e^{2t}$$

Question 5. Solve the initial value problem.

$$y' - 5y = -25t, \qquad y(0) = 7$$

(i) Associated homog DE: $y' - 5y = 0 \Rightarrow$ auxiliary equation $r - 5 = 0$

$\Rightarrow$ general solution of homog DE is $y = ce^{5t}$.

(ii) Find a particular solution of the nonhomog DE. Try $y = At + B$, $y' = A$

$$\Rightarrow \underbrace{A}_{y'} - 5\underbrace{(At + B)}_{y} = -25t$$

$$\Rightarrow A - 5At - 5B = -25t$$

$$\Rightarrow (-5A)t + (A - 5B) = (-25)t + 0$$

$$\Rightarrow \left.\begin{aligned} -5A &= -25 \\ A - 5B &= 0 \end{aligned}\right\} \Rightarrow A = 5,\ B = 1.$$

A particular soln of the nonhomog DE is $y = 5t + 1$.

(i) & (ii) $\Rightarrow$ the general soln of the nonhomog DE is $y = ce^{5t} + 5t + 1$.

(iii) Use initial condition to find c.

$$7 = y(0) = c \cdot \underbrace{e^0}_{1} + \underbrace{5 \cdot 0}_{0} + 1 \Rightarrow 7 = c + 1$$

$$c = 6$$

Final answer: $y = 6e^{5t} + 5t + 1$.

Question 6. Find the **general** solution of the differential equation.

$$y'' + 4y = 6\sin t + 6\cos t$$

(i) Associated homog DE: $y'' + 4y = 0$ Aux eqn $r^2 + 4 = 0 \Rightarrow r^2 = -4$

$\Rightarrow r = \pm\sqrt{-4} = \pm 2i$ or $0 \pm 2i$. $e^{0t}\sin 2t$, $e^{0t}\cos 2t$

General soln of homog DE is $y = c_1 \sin 2t + c_2 \cos 2t$.

(ii) Find a particular soln of the nonhomog DE.

Try $y = A\sin t + B\cos t \Rightarrow y' = A\cos t - B\sin t$

$\Rightarrow y'' = -A\sin t - B\cos t$. Plug these into the DE.

$$\underbrace{-A\sin t - B\cos t}_{y''} + 4(\underbrace{A\sin t + B\cos t}_{y}) = 6\sin t + 6\cos t$$

$$(-A + 4A)\sin t + (-B + 4B)\cos t = 6\sin t + 6\cos t$$

$$3A\sin t + 3B\cos t = 6\sin t + 6\cos t$$

$\Rightarrow A = 2, B = 2$. Particular soln is $2\sin t + 2\cos t$.

Final answer: $y = c_1 \sin 2t + c_2 \cos 2t + 2\sin t + 2\cos t$

Question 7. Solve the initial value problem.

$$y' - y = 6, \qquad y(0) = 1$$

Associated homog DE: $y' - y = 0$ Auxiliary eqn $r - 1 = 0 \Rightarrow r = 1$
$\Rightarrow$ general soln of homog DE is $y = Ce^t$.

Particular soln of nonhomog DE? Try $y = A \Rightarrow y' = 0$.
Plug into DE: $\underbrace{0}_{y'} - \underbrace{A}_{y} = 6 \Rightarrow A = -6$ $y = -6$ is a particular soln

So <u>general</u> soln to <u>nonhomog</u> DE is $y = Ce^t - 6$.

Use initial condition to find C.

$$1 = y(0) = C \cdot \underbrace{e^0}_{1} - 6 \Rightarrow 1 = C - 6 \Rightarrow C = 7$$

Final answer: $y = 7e^t - 6$

Question 8. Find a **particular** solution of the differential equation.

$$y'' + y = \sec t$$

Associated homog DE: $y'' + y = 0$. We know two independent solns are $y_1 = \sin t$ and $y_2 = \cos t$. So $y_1'' + y_1 = 0$, $y_2'' + y_2 = 0$.

Try a soln of the form $y = v_1 y_1 + v_2 y_2$

$\Rightarrow y' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$ Impose the condition $v_1' y_1 + v_2' y_2 = 0$ (i)

$y' = v_1 y_1' + v_2 y_2'$

$y'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$ Plug into DE:

$$\underbrace{v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''}_{y''} + \underbrace{v_1 y_1 + v_2 y_2}_{y} = \sec t$$

$$v_1' y_1' + v_1 \underbrace{(y_1'' + y_1)}_{0} + v_2' y_2' + v_2 \underbrace{(y_2'' + y_2)}_{0} = \sec t$$

$v_1' y_1' + v_2' y_2' = \sec t$ (ii) Then (i) $v_1' \sin t + v_2' \cos t = 0$

(ii) $v_1' \cos t - v_2' \sin t = \sec t$

(i) × $\sin t$, (ii) × $\cos t$ $\Rightarrow$

$$\begin{aligned} v_1' \sin^2 t + v_2' \sin t \cos t &= 0 \\ v_1' \cos^2 t - v_2' \sin t \cos t &= 1 \\ \hline v_1'(\sin^2 t + \cos^2 t) &= 1 \end{aligned}$$

$v_1' = 1 \Rightarrow \sin t + v_2' \cos t = 0$

$v_1 = t \quad \Rightarrow v_2' = -\tan t$

$v_2 = \ln \cos t$

A particular soln is $y = t \sin t + \cos t \cdot \ln \cos t$

Question 9. Find a particular solution of the differential equation using any correct method.

$$y'' - 2y' + y = t^{-3}e^t$$

(Hint: Try a solution of the form $y = At^{-1}e^t$.)

METHOD 1. Follow the hint. $y = At^{-1}e^t$

$\Rightarrow y' = A(t^{-1})'e^t + At^{-1}(e^t)' = -At^{-2}e^t + At^{-1}e^t$

$\Rightarrow y'' = -A(t^{-2})'e^t - At^{-2}(e^t)' + A(t^{-1})'e^t + At^{-1}(e^t)'$

$= 2At^{-3}e^t - At^{-2}e^t - At^{-2}e^t + At^{-1}e^t$

$= 2At^{-3}e^t - 2At^{-2}e^t + At^{-1}e^t$

Plug these into the DE:

$$\underbrace{(2At^{-3}e^t - 2At^{-2}e^t + At^{-1}e^t)}_{y''} - 2\underbrace{(-At^{-2}e^t + At^{-1}e^t)}_{y'} + \underbrace{(At^{-1}e^t)}_{y} = t^{-3}e^t$$

$$2At^{-3}e^t - 2At^{-2}e^t + At^{-1}e^t + 2At^{-2}e^t - 2At^{-1}e^t + At^{-1}e^t = t^{-3}e^t$$

(The two terms containing $t^{-2}e^t$ sum to 0, and the three terms containing $t^{-1}e^t$ sum to 0!)

$$2At^{-3}e^t = t^{-3}e^t \quad \Rightarrow 2A = 1 \quad \Rightarrow A = \frac{1}{2}$$

A particular solution is $y = \frac{1}{2}t^{-1}e^t$.

Question 9. Find a particular solution of the differential equation using any correct method.

$$y'' - 2y' + y = t^{-3}e^t$$

(Hint: Try a solution of the form $y = At^{-1}e^t$.)

METHOD 2 (ignores hint) Associated homog DE is $y'' - 2y' + y = 0$.

Auxiliary eqn is $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$ is a double root

$\Rightarrow$ two independent solns of the homog DE are $y_1 = e^t$ and $y_2 = te^t$.

Try to find a particular soln of the nonhomog of the form $y = v_1 y_1 + v_2 y_2$

$\Rightarrow y' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$ Impose the condition

$v_1' y_1 + v_2' y_2 = 0 \quad (i)$

$$y' = v_1 y_1' + v_2 y_2' \Rightarrow y'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

Plug into the nonhomog DE:

$$v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'' \underbrace{-2(v_1 y_1' + v_2 y_2')}_{-2v_1 y_1' - 2v_2 y_2'} + v_1 y_1 + v_2 y_2 = t^{-3}e^t$$

$v_1 y_1'' - 2v_1 y_1' + v_1 y_1 = 0$

$v_2 y_2'' - 2v_2 y_2' + v_2 y_2 = 0$

$$v_1' y_1' + v_2' y_2' = t^{-3}e^t \quad (ii)$$

$(i) \Rightarrow v_1' e^t + v_2' te^t = 0$

$(ii) \Rightarrow v_1' e^t + v_2' \cdot (e^t + te^t) = t^{-3}e^t$

$\Rightarrow (ii) - (i): v_2' \cdot e^t = t^{-3}e^t$

$\Rightarrow v_2' = t^{-3}$. Plug into (i)

$\Rightarrow v_1' = -t^{-2}$

$$\Rightarrow v_1 = t^{-1}, \quad v_2 = \frac{t^{-2}}{-2} \Rightarrow y = t^{-1}e^t + \frac{t^{-2}}{-2} te^t$$

$$= t^{-1}e^t - \frac{1}{2}t^{-1}e^t = \frac{1}{2}t^{-1}e^t$$