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MAP 2302 Test 3 Tuesday November 19th
Total possible score: 18 points

Question 1. Consider the differential equation $x^2y'' + 3xy' + y = 0$. By substituting a proposed solution of the form $y = x^r$ (and its derivatives), show that r must be -1.

$$y = x^r \Rightarrow y' = rx^{r-1} \Rightarrow y'' = r(r-1)x^{r-2}$$

Plug these into the D.E.

$$x^2 \cdot \underbrace{r(r-1)x^{r-2}}_{y''} + 3x \cdot \underbrace{rx^{r-1}}_{y'} + \underbrace{x^r}_{y} = 0$$

x^r is common factor

$$x^r \cdot \left(\underbrace{r(r-1)}_{r^2-r} + 3r + 1 \right) = 0$$

$$r^2 - r + 3r + 1 = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r + 1 = 0 \Rightarrow r = -1$$

Question 2. Find the Laplace transform of $e^{3t}(\cos\sqrt{10}t + 2t^3)$.

$$f(t) = \underbrace{e^{3t}\cos\sqrt{10}\,t}_{\substack{\text{of the form} \\ e^{at}\cos bt \\ \text{with } a=3,\ b=\sqrt{10}}} + \underbrace{2t^3 e^{3t}}_{\substack{\text{of the form} \\ e^{at}t^n \\ \text{with } a=3,\ n=3}}$$

$$\mathcal{L}\{f(t)\} = \frac{s-3}{(s-3)^2+(\sqrt{10})^2} + 2\cdot\frac{3!}{(s-3)^4}$$

or

$$\frac{s-3}{s^2-6s+\underbrace{9+10}_{19}} + \frac{12}{(s-3)^4}$$

Question 3. Find the inverse Laplace transform of $\frac{3s+43}{s^2+2s+101}$.

$$F(s) = \frac{3s+43}{s^2+2s+1+100} = \frac{3s+43}{(s+1)^2+10^2}$$

$$= \frac{3s+3}{(s+1)^2+10^2} + \frac{40}{(s+1)^2+10^2}$$

$$= 3\cdot\underbrace{\frac{s+1}{(s+1)^2+10^2}}_{\substack{\text{of the form}\\ \frac{s-a}{(s-a)^2+b^2}\\ a=-1,\ b=10}} + 4\cdot\underbrace{\frac{10}{(s+1)^2+10^2}}_{\substack{\text{of the form}\\ \frac{b}{(s-a)^2+b^2}\\ a=-1,\ b=10}}$$

$$\mathcal{L}^{-1}\{F(s)\} = 3\cdot e^{-t}\cos 10t + 4\cdot e^{-t}\sin 10t$$

Question 4. Find the partial fraction expansion of $\frac{2s^3+2s^2+2s+1}{s^4+s^3}$.

Denominator factors as $s^3 \cdot (s+1)$

$$\frac{2s^3+2s^2+2s+1}{s^4+s^3} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+1}$$

$\downarrow$ multiply both sides by $s^3(s+1)$

$$2s^3+2s^2+2s+1 = As^2(s+1) + Bs(s+1) + C(s+1) + Ds^3$$

$s=0$: $\quad 1 = 0 + 0 + C\cdot 1 + 0 \Rightarrow C=1$

$s=-1$: $\quad \underbrace{-2+2-2+1}_{-1} = 0 + 0 + 0 + D\cdot(-1) \Rightarrow D=1$

$$\begin{aligned} 2s^3+2s^2+2s+1 &= As^2(s+1) + Bs(s+1) + (s+1) + s^3 \\ &= As^3 + As^2 + Bs^2 + Bs + s + 1 + s^3 \\ &= (A+1)s^3 + (A+B)s^2 + (B+1)s + 1 \end{aligned}$$

$$\Rightarrow \left.\begin{aligned} A+1 &= 2 \\ A+B &= 2 \\ B+1 &= 2 \end{aligned}\right\} \Rightarrow A=1,\ B=1$$

Answer: $\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s+1}$

Question 5. Find the inverse Laplace transform of $\dfrac{2s^2+14s+8}{(s-3)(s^2+10s+29)}$.

Partial fractions: $\dfrac{2s^2+14s+8}{(s-3)(s^2+10s+29)} = \dfrac{A}{s-3} + \dfrac{Bs+C}{s^2+10s+29}$

$$\begin{aligned}
\Rightarrow 2s^2+14s+8 &= A(s^2+10s+29) + (Bs+C)(s-3)\\
&= As^2 + 10As + 29A + Bs^2 - 3Bs + Cs - 3C\\
&= (A+B)s^2 + (10A-3B+C)s + (29A-3C)
\end{aligned}$$

$$\Rightarrow \left.\begin{array}{l} A+B = 2 \\ 10A-3B+C = 14 \end{array}\right\} \Rightarrow \begin{array}{l} 3A+3B = 6 \\ 10A-3B+C = 14 \\ \hline 13A \quad +C = 20 \end{array} \nearrow \begin{array}{l} 39A+3C = 60 \\ 29A-3C = 8 \\ \hline 68A \quad = 68 \end{array} \Rightarrow A = 1$$

$$29A \quad -3C = 8$$

$$\Rightarrow B = 1,\ C = 7$$

$$F(s) = \frac{1}{s-3} + \frac{s+7}{s^2+10s+29} = \frac{1}{s-3} + \frac{s+7}{\underbrace{(s+5)^2}_{s^2+10s+25} + 2^2}$$

$$= \frac{1}{s-3} + \frac{s+5}{(s+5)^2+2^2} + \frac{2}{(s+5)^2+2^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{3t} + e^{-5t}\cos 2t + e^{-5t}\sin 2t$$

Question 6. Find $\mathcal{L}^{-1}\{F\}$, given $s^2F(s) - 4F(s) = \dfrac{5}{s+1}$.

$$(s^2-4)F(s) = \frac{5}{s+1}$$

$$F(s) = \frac{5}{(s+1)(s^2-4)} = \frac{5}{(s+1)(s+2)(s-2)}$$

$$\frac{5}{(s+1)(s+2)(s-2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-2}$$

$$5 = A(s+2)(s-2) + B(s+1)(s-2) + C(s+1)(s+2)$$

$s=-1$: $5 = A\cdot 1\cdot(-3) + 0 + 0$

$s=-2$: $5 = 0 + B\cdot(-1)(-4) + 0$

$s=2$: $5 = 0 + 0 + C\cdot 3\cdot 4$

$5 = -3A$
$A = -\frac{5}{3}$

$5 = 4B$
$B = \frac{5}{4}$

$5 = 12C$
$C = \frac{5}{12}$

$$F(s) = -\frac{5}{3}\cdot\frac{1}{s+1} + \frac{5}{4}\cdot\frac{1}{s+2} + \frac{5}{12}\cdot\frac{1}{s-2}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{5}{3}e^{-t} + \frac{5}{4}e^{-2t} + \frac{5}{12}e^{2t}$$

Question 7. Solve the initial value problem using Laplace transforms.

$$y'' + 2y' + 10y = 0, \qquad y(0) = 1, \qquad y'(0) = 5$$

$$\mathcal{L}\{y'' + 2y' + 10y\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 10\mathcal{L}\{y\} = 0$$

$$s^2F(s) - s\cdot \underbrace{y(0)}_{1} - \underbrace{y'(0)}_{5} + 2\big(sF(s) - \underbrace{y(0)}_{1}\big) + 10F(s) = 0$$

$$s^2F(s) - s - 5 + 2sF(s) - 2 + 10F(s) = 0$$

$$s^2F(s) + 2sF(s) + 10F(s) = s+7$$

$$(s^2 + 2s + 10)F(s) = s+7$$

$$F(s) = \frac{s+7}{s^2+2s+10} = \frac{s+7}{s^2+2s+1+9} = \frac{s+7}{(s+1)^2+3^2}$$

$$= \frac{s+1}{(s+1)^2+3^2} + 2\cdot\frac{3}{(s+1)^2+3^2}$$

$$\mathcal{L}^{-1}\{F(s)\} = e^{-t}\cos 3t + 2e^{-t}\sin 3t$$

Question 8. For the given initial value problem, find $Y(s)$ (the Laplace transform of the solution) but do not bother finding the solution.

$$y'' - 7y' + 6y = -50te^t, \qquad y(0) = 3, \qquad y'(0) = 10$$

$$\mathcal{L}\{y'' - 7y' + 6y\} = \mathcal{L}\{-50te^t\}$$

$$\mathcal{L}\{y''\} - 7\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = -50\,\mathcal{L}\{te^t\}$$

$$s^2Y(s) - s\cdot\underbrace{y(0)}_{3} - \underbrace{y'(0)}_{10} - 7\big(sY(s) - \underbrace{y(0)}_{3}\big) + 6Y(s) = -50\cdot\frac{1}{(s-1)^2}$$

$$s^2Y(s) - 3s - 10 - 7sY(s) + 21 + 6Y(s) = \frac{-50}{(s-1)^2}$$

$$s^2Y(s) - 7sY(s) + 6Y(s) = \frac{-50}{(s-1)^2} + 3s - 11$$

$$(s^2 - 7s + 6)Y(s) = \frac{-50}{(s-1)^2} + 3s - 11$$

$$Y(s) = \frac{-50}{(s-1)^2(s^2-7s+6)} + \frac{3s-11}{s^2-7s+6}$$ DONE

(BTW, $s^2 - 7s + 6$ factors as $(s-1)(s-6)$)

Question 9. Consider the differential equation $x^2y'' + 3xy' + \overbrace{(1+x)y}^{y+xy} = 0$. If x is near 0, this resembles the equation in Question 1. Find the first five terms of a solution that looks like x^{-1} times a power series.

$$y = x^{-1}\cdot\left(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots\right)$$

$$y = a_0x^{-1} + a_1 + a_2x + a_3x^2 + a_4x^3 + \dots$$

$$y' = -a_0x^{-2} + a_2 + 2a_3x + 3a_4x^2 + \dots$$

$$y'' = 2a_0x^{-3} + 2a_3 + 6a_4x + \dots$$

$$x^2y'' = 2a_0x^{-1} + 0 + 0 + 2a_3x^2 + 6a_4x^3 + \dots$$

$$3xy' = -3a_0x^{-1} + 0 + 3a_2x + 6a_3x^2 + 9a_4x^3 + \dots$$

$$y = a_0x^{-1} + a_1 + a_2x + a_3x^2 + a_4x^3 + \dots$$

$$xy = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$0 = 0 + (a_1+a_0) + (4a_2+a_1)x + (9a_3+a_2)x^2 + (16a_4+a_3)x^3 + \dots$$

$$\left.\begin{aligned}
a_1 + a_0 = 0 &\Rightarrow a_1 = -a_0 \\
4a_2 + a_1 = 0 &\Rightarrow a_2 = -\tfrac{1}{4}a_1 = +\tfrac{1}{4}a_0 \\
9a_3 + a_2 = 0 &\Rightarrow a_3 = -\tfrac{1}{9}a_2 = -\tfrac{1}{36}a_0 \\
16a_4 + a_3 = 0 &\Rightarrow a_4 = -\tfrac{1}{16}a_3 = +\tfrac{1}{576}a_0
\end{aligned}\right\} \Rightarrow y = a_0x^{-1} - a_0 + \frac{1}{4}a_0x - \frac{1}{36}a_0x^2 + \frac{1}{576}a_0x^3 + \dots$$