What is a differential equation?

An equation with an unknown *function* that contains derivatives of the function.

To find the function is to 'solve' the differential equation.

Some examples of differential equations:

$$f'(x) = f(x)$$
$$f'(x) = 5f(x)$$
$$xf'(x) = 3f(x)$$
$$f''(x) = -f(x)$$

How do we **solve** a differential equation?

Maybe in some cases, we can *guess* a solution!

You should already know some common functions and their derivatives.

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$
$$\frac{d}{dx}\left(e^x\right) = e^x$$
$$\frac{d}{dx}\left(\sin x\right) = \cos x$$
$$\frac{d}{dx}\left(\cos x\right) = -\sin x$$

Can any of these derivative facts help you *guess* solutions to the differential equations above?

WEEK 1 EXAMPLE 1:
Determine whether $y = x^3$ is a solution to the differential equation $xy' = 3y$.

If we're concerned with functions and their derivatives, then we have an **independent variable** ('input') and a **dependent variable** ('output').

Often (but not always), the independent variable is x and the dependent variable is y.

How are the following two differential equations different? Can you solve them?

$$y' = 5x$$

$$y' = 5y$$

How are the following two differential equations different? Can you solve them?

$$y' = \cos x$$

$$y' = \cos y$$

Ordinary differential equations and partial differential equations

In this course, our functions will have just **one** independent variable.

If we know there's only one (say x) then we can just write y' rather than $y'(x)$ or $\dfrac{dy}{dx}$.

If a differential equation involves only one independent variable, then it's called an **ordinary** differential equation. So in this course, we work with ordinary differential equations.

In mathematics (and applications of mathematics) it's also common to have functions of more than one variable.

For example, the temperature of a metal rod (f) could depend on the position within the rod (x) as well as time (t).

If f is a function of x **and** t, then we can consider the derivatives with respect to **either** of those variables (**partial** derivatives).

One example of a partial differential equation is the **heat equation**: $\dfrac{\partial f}{\partial t} = \dfrac{\partial^2 f}{\partial x^2}$

We will not study partial *differential equations* in this course, but partial *derivatives* will make a brief appearance when we study exact differential equations in Week 2.

The 'order' of a differential equation

We sometimes refer to y'', y''', ... as the *higher order* derivatives.

The highest order of a derivative appearing in a differential equation is called the **order** of the differential equation. For example:

$$y' = 5y \qquad \text{is first order}$$

$$y' = \frac{1}{x^2+1} \qquad \text{is first order}$$

$$y'' = -y \qquad \text{is second order}$$

$$y'' - 5y' + 6y = 0 \qquad \text{is second order}$$

Linear and nonlinear differential equations

A differential equation is called **linear** if we have only linear functions of the **dependent** variable and its derivatives. (The coefficients can be **any** functions of the **independent** variable.) For example:

$$y'' + y^3 = 0 \qquad \text{is } \textbf{not} \text{ linear, because of } y^3$$

$$x^3 y' = x^3 + y \qquad \textbf{is} \text{ linear, despite } x^3 \text{ (linear in } y \text{ and } y')$$

$$y'' = \sin y \qquad \text{is } \textbf{not} \text{ linear, because of } \sin y$$

$$y'' + yy' = e^x \qquad \text{is } \textbf{not} \text{ linear, because of } yy'$$

Solutions to differential equations

A solution to a differential equation is a function that satisfies it.

An **explicit** solution to an ordinary differential equation is a function of the form $y = f(x)$ that satisfies it.

IMPORTANT: Even if it's not obvious how to **find** a solution, it is straightforward to **verify** whether something is a solution.

WEEK 1 EXAMPLE 2: Consider the differential equation given below.

$$y'' - 5y' + 6y = 0$$

Which of the following functions are solutions?

$$y = e^x$$

$$y = e^{2x}$$

$$y = e^{3x}$$

$$y = 7e^{2x} + 11e^{3x}$$

$$y = \sin x + \cos x$$

Implicit functions and implicit solutions

In Calculus I, you study **implicit differentiation**. (Section 3.8 of Briggs)

This is because we sometimes have equations that **implicitly** define y as a function of x.

For example, $y^3 + y = x$. (Can't rearrange to solve for y explicitly)

This still meaningfully defines y as a function of x. (If $x = 2$, then $y = 1$, and if x increases, then y will increase)

WEEK 1 EXAMPLE 3: The equation $y^2 + xy = 8$ implicitly defines y as a function of x.

Verify that this function is a solution of the differential equation

$$y' = \frac{-y}{x + 2y}$$

(What happens if we replace 8 with a different constant?)

Families of solutions

WEEK 1 EXAMPLE 4: Consider the following differential equation.

$$y'' - 5y' + 6y = 0$$

(i) Show that any function of the form $y = Ce^{3x}$ is a solution, where C is a constant. (This is a 'one-parameter family' of solutions.)

(ii) Show that any function of the form $y = C_1e^{2x} + C_2e^{3x}$ is a solution, where C_1 and C_2 are constants. (This is a 'two-parameter family' of solutions.)

Initial value problems

A first-order initial value problem consists of a first-order differential equation (involving y' and y) together with an initial condition of the form $y(x_0) = y_0$.

An example of an initial value problem:

$$y' = x - y, \qquad y(0) = 1$$

Another example of an initial value problem:

$$y' = -\frac{x}{y}, \qquad y(1) = 4$$

Both of these (first-order) initial value problems have the form

$$y' = f(x, y), \qquad y(x_0) = y_0$$

Important theoretical fact: If f and $\partial f/\partial y$ are continuous on some rectangle containing (x_0, y_0), then the initial value problem is guaranteed to have a unique solution on some interval containing x_0.

Numerically approximating a solution

Let's explore how we might **approximate** the solution to the initial value problem

$$y' = x - y, \qquad y(0) = 1.$$

Remember, $y' = \dfrac{dy}{dx}$ is the **slope** of the graph of y.

Currently, y is an **unknown** function of x, say $y = g(x)$.

We want to find a formula for $g(x)$, or at least approximate it.

Whatever the graph of $g(x)$ looks like, the slope at any point is equal to $x - y$.

So for example, **if** the graph goes through the point $(7, 4)$, the slope at that point would be $7 - 4 = 3$.

For 'every' point in the plane (or many points), we can draw what the slope **would** be at that point. This gives us a **direction field** for the differential equation.

Euler's method for approximating solutions

Again consider the following initial value problem.

$$y' = x - y, \qquad y(0) = 1.$$

The solution is **some** function $y = g(x)$.

The differential equation says that the slope at any point is equal to $x - y$.

Remember, slope $= y' = \dfrac{dy}{dx} \approx \dfrac{\Delta y}{\Delta x}$

If x increases by Δx, the change in y is approximately $(x - y)\Delta x$.

If we try to **approximate** the solution using $\Delta x = 0.1$, we can generate the following table.

x	y	$x - y$	$\Delta y = (x - y)\Delta x$
0	1	$0 - 1 = -1$	-0.1
0.1	0.9	$0.1 - 0.9 = -0.8$	-0.08
0.2	0.82	$0.2 - 0.82 = -0.62$	-0.062
0.3	0.758	$0.3 - 0.758 = -0.458$	-0.0458
0.4	0.71something		

By the way, this initial value problem **can** be solved explicitly. The solution is $y = x + 2e^{-x} - 1$