Separable differential equations

These are a special type of first-order differential equation.

If we have an equation of the form

$$y' = f(x, y),$$

then it **might** be the case that f is a function of x times a function of y. If so, then we can 'separate' the variables and rewrite in the form

$$h(y)\frac{dy}{dx} = g(x)$$

and then 'separate' the variables

$$h(y)\,dy = g(x)\,dx$$

and then integrate each side individually.

WEEK 2 EXAMPLE 1: Solve the differential equation.

$$y' = x^2 y$$

WEEK 2 EXAMPLE 2: Solve the differential equation.

$$y' = \frac{x}{y^2\sqrt{1+x}}$$

Remember, a differential equation can have **infinitely** many solutions.

But an *initial value problem* will (usually) have a **unique** solution.

WEEK 2 EXAMPLE 3: Solve the initial value problem.

$$\frac{dy}{dx} = (1 + y^2) \tan x, \qquad y(0) = \sqrt{3}$$

First-order linear equations

A first-order linear differential equation can be written in 'standard form'

$$y' + P(x)y = Q(x)$$

How can we find a general technique for solving this?

Key: The left side resembles the result of performing the product rule.

Maybe we can multiply both sides by some function $\mu(x)$

$$\mu(x)y' + \mu(x)P(x)y = \mu(x)Q(x)$$

and have the left side be **exactly** the result of the product rule.

This would work if we have $\mu(x)P(x) = \mu'(x)$.

Equivalently, we want

$$P(x) = \frac{1}{\mu(x)}\mu'(x).$$

The right-hand side is the derivative of $\ln\big(\mu(x)\big)$.

So we have the following steps:

$$\begin{aligned} P(x) &= \frac{1}{\mu(x)}\mu'(x) \\ P(x) &= \frac{d}{dx}\Big(\ln\big(\mu(x)\big)\Big) \\ \int P(x)\,dx &= \ln\big(\mu(x)\big) \\ e^{\int P(x)dx} &= \mu(x) \end{aligned}$$

CONCLUSION: If you have a first-order linear differential equation

$$y' + P(x)y = Q(x)$$

and you multiply both sides by the 'integrating factor'

$$\mu(x) = e^{\int P(x)\,dx}$$

then you'll get an equation of the form

$$\mu(x)y' + \mu'(x)y = \mu(x)Q(x)$$

which we can then (probably? maybe?) solve.

WEEK 2 EXAMPLE 4: Find the general solution to the differential equation.

$$y' - y - e^{3x} = 0$$

We can also have an initial value problem involving a first-order linear equation.

WEEK 2 EXAMPLE 5: Solve the initial value problem.

$$y' + 4y - e^{-x} = 0, \qquad y(0) = \frac{4}{3}$$

Exact differential equations

We saw that when solving separable differential equations, we sometimes treat dy and dx as separate entities.

This is not always 100% mathematically rigorous, but can still be useful in practice and is often done in physics.

Informal idea: x changes by a small amount dx, and y changes by a small amount dy.

Remember that an equation of the form $F(x, y) = C$ can define a curve.

It also gives an implicit relationship between x and y.

One example: $x^2 + y^2 = 4$ (circle with radius 2)

Another example: $x^2 + xy + y^2 = 3$ (more complicated curve)

In Calculus I, you learn to apply implicit differentiation to equations like these.

In the first example on the previous page, if you take $\dfrac{d}{dx}$ of both sides, you get

$$\frac{d}{dx}\left(x^2+y^2\right)=\frac{d}{dx}\left(4\right)$$
$$2x+2y\cdot y'=0$$
$$\text{or}\qquad 2x+2y\cdot\frac{dy}{dx}=0$$

This could also be written $2x\,dx+2y\,dy=0$.

Notice we would get exactly the same result if the 4 on the right side was a different constant.

This means that if we're **given** the differential equation $2x\,dx+2y\,dy=0$, a one-parameter family of solutions is $x^2+y^2=C$.

In the second example, we would get

$$\frac{d}{dx}\left(x^2+xy+y^2\right)=\frac{d}{dx}\left(3\right)$$
$$(x^2)'+(x)'y+x(y)'+(y^2)'=(3)'$$
$$2x+1\cdot y+x\cdot\frac{dy}{dx}+2y\cdot\frac{dy}{dx}=0$$
$$\text{or}\qquad 2x\,dx+y\,dx+x\,dy+2y\,dy=0$$

This could also be written $(2x+y)dx+(x+2y)dy=0$.

So, if we're **given** the differential equation $(2x+y)dx+(x+2y)dy=0$, a one-parameter family of solutions is $x^2+xy+y^2=C$. (Again, note that the 3 on the right-hand side could have been any constant.)

Notice that if $F(x, y) = x^2 + xy + y^2$, then

$$\frac{\partial}{\partial x}\left(x^2 + xy + y^2\right) = 2x + y$$
$$\frac{\partial}{\partial y}\left(x^2 + xy + y^2\right) = x + 2y$$

so the differential equation $(2x + y)dx + (x + 2y)dy = 0$ has the form

$$\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = 0.$$

In general, if we have a differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0 \tag{1}$$

and there exists a function $F(x, y)$ with the property that

$$\frac{\partial F}{\partial x} = M(x, y) \qquad \text{and} \qquad \frac{\partial F}{\partial y} = N(x, y)$$

then the differential equation (1) is called **exact**, and a one-parameter family of solutions to the differential equation is

$$F(x, y) = C.$$

How do we tell whether a differential equation is exact?

FACT: If $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then equation (1) is exact.

Why? Because $\frac{\partial M}{\partial y}$ would be $\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$ and $\frac{\partial N}{\partial x}$ would be $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$.

WEEK 2 EXAMPLE 6: Solve the differential equation.

$$(2x + y)\,dx + (x - 2y)\,dy = 0$$