Special integrating factors

If a differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is **not** exact (i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$) then maybe we can find a function $\mu(x,y)$ such that

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact. But if we try to make this work, it turns out that we have to solve a *partial* differential equation!

We can instead try to find an integrating factor μ that's either a function of x only, or a function of y only. If we play around with the algebra that's involved, we eventually get the following.

- If $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})/N$ depends only on x, try $\mu = \mu(x)$.
- If $(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})/M$ depends only on y, try $\mu = \mu(y)$.

WEEK 3 EXAMPLE 1: Consider the differential equation

$$\big(2xy\big)\,dx + \big(y^2 - 3x^2\big)\,dy = 0$$

Is it exact? If it's not exact, can we find an integrating factor?

WEEK 3 EXAMPLE 2: Solve the differential equation.

$$(3x^2 + y)\,dx + (x^2y - x)\,dy = 0$$

HINT: Multiply both sides by x^{-2}.

Homogeneous equations

If a differential equation of the form

$$\frac{dy}{dx} = f(x, y)$$

has a right-hand side that's a function of y/x, then the equation is called **homogeneous**.

To solve a homogeneous differential equation, substitute $v = y/x$ (so $y = vx$).

By the product rule, we have $\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \cdot 1$.

The substitution $v = y/x$ will turn the homogeneous equation into a *separable* equation.

WEEK 3 EXAMPLE 3: Solve the differential equation.

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}$$

(Side note: This differential equation can be rearranged as $(y^2 - xy)dx + x^2 dy = 0$. Does that help? Are there any other ways to solve this equation?)

Substitutions and transformations

Equations of the form $\frac{dy}{dx} = G(ax + by)$

Key: The substitution $z = ax + by$ will turn this into a separable equation.

WEEK 3 EXAMPLE 4: Solve the differential equation.

$$\frac{dy}{dx} = \sin(x - y)$$

Bernoulli equations

Any differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a Bernoulli equation.

It turns out that if we multiply by y^{-n} and then substitute $v = y^{1-n}$, we will get a linear differential equation.

WEEK 3 EXAMPLE 5: Solve the differential equation.

$$\frac{dy}{dx} - y = e^{2x}y^3$$

Mathematical models involving first-order DEs

For example, suppose we want to model population growth.

Simplest model: Rate of change of population is proportional to current population.

$$\frac{dP}{dt} = K \cdot P$$

(e.g. bacteria in a petri dish, with plenty of food and space)

The solution to that differential equation will be unlimited exponential growth.

For a more nuanced model, consider that there could be interactions between individuals that contribute negatively to population growth. (War, contagious diseases, competition for limited food and space. . .)

Maybe population can be modeled by a differential equation of the form

$$\frac{dP}{dt} = A \cdot P - B \cdot P^2$$

where A and B are constants.

As a specific numerical example, consider

$$\frac{dP}{dt} = \frac{1}{4}P - \frac{1}{4000}P^2$$

What kind of differential equation is this? How would we solve it?