The Mass-Spring Oscillator

Consider a mass on a spring, with one end of the spring attached to a wall.

Let y be the position of the mass relative to its equilibrium position.

There are various forces acting on the mass.

One is the force due to the spring itself, which by Hooke's law is opposite to the current displacement:

$$F_{\text{spring}} = -ky$$

There is also friction. For vibrational motion, this can often be modeled as proportional to velocity:

$$F_{\text{friction}} = -b\frac{dy}{dt} = -by'$$

There could also be further external forces that might depend on time (e.g. maybe I reach in and tap the mass every 30 seconds, or something).

$$F_{\text{ext}} = F_{\text{ext}}(t)$$

Newton's second law says that force is mass times acceleration.

The total force acting on the mass is

$$\begin{aligned} ma &= F_{\text{spring}} + F_{\text{friction}} + F_{\text{ext}} \\ my'' &= -ky - by' + F_{\text{ext}}(t) \\ my'' + by' + ky &= F_{\text{ext}}(t) \end{aligned}$$

This is a second-order linear differential equation.

Special cases:

What if there's no friction and no external force?

Then $b = 0$ and $F_{\text{ext}}(t) = 0$.

The differential equation becomes

$$my'' + ky = 0$$

so for example, maybe $5y'' + 3y = 0$.

Can we **guess** a solution to the equation $5y'' + 3y = 0$?

Can you use physics **or** math to guess a type of function that might work?

Special cases:

What if there **is** friction, but still no external force?

$$my'' + by' + ky = 0$$

For example, maybe $1y'' + 4y' + 13y = 0$.

Physically speaking, what kind of function might describe the behavior of the spring?

Homogeneous Linear Equations

Any differential equation of the form

$$ay'' + by' + cy = 0$$

is called a **homogeneous** linear differential equation.

('Homogeneous' in this context means that the right side is 0.)

How do we find solutions to a homogeneous equation?

We know that exponential functions have nice derivatives.

Suppose $y = e^{rt}$, where r is a constant. We then have

$$y = e^{rt}$$
$$y' = re^{rt}$$
$$y'' = r^2e^{rt}$$

What happens if we plug these into $ay'' + by' + cy = 0$?

If we try $y = e^{rt}$ in the equation $ay'' + by' + cy = 0$, we get

$$\begin{aligned} a \cdot r^2 e^{rt} + b \cdot r e^{rt} + c \cdot e^{rt} &= 0 \\ (ar^2 + br + c)e^{rt} &= 0 \end{aligned}$$

Since e^{rt} is positive, this means we must have $ar^2 + br + c = 0$.

This is an **algebraic** equation that we can solve for r.

(The equation $ar^2+br+c = 0$ is called the 'auxiliary equation' associated with the differential equation.)

WEEK 4 EXAMPLE 1: Find two solutions of the differential equation.

$$2y'' + 7y' - 4y = 0$$

It turns out that the functions

$$y_1 = e^{t/2} \qquad \text{and} \qquad y_2 = e^{-4t}$$

are two different solutions to the differential equation $2y'' + 7y' - 4y = 0$.

More specifically, y_1 and y_2 are two **linearly independent** solutions (for two functions, this means neither is a constant multiple of the other).

Can we find more solutions? Sums or multiples of existing solutions?

In fact, it turns out that any 'linear combination'

$$y = c_1y_1 + c_2y_2 = c_1e^{t/2} + c_2e^{-4t}$$

is a solution. This is the **general solution** to our homogeneous linear differential equation.

If you're solving an initial value problem, then two initial values, e.g. $y(0)$ and $y'(0)$, will enable you to find values for c_1 and c_2.

WEEK 4 EXAMPLE 2: Solve the initial value problem.

$$y'' + 2y' - 8y = 0, \qquad y(0) = 3, \qquad y'(0) = -12$$

Repeated roots

What happens if the auxiliary equation has a repeated root?

$$y'' + 6y' + 9y = 0$$

The auxiliary equation is $r^2 + 6r + 9 = 0$, or equivalently, $(r+3)^2 = 0$. So $y = e^{-3t}$ is one solution.

How do we find two **independent** solutions, or a **two-parameter family** of solutions, in order to formulate our general solution?

If the auxiliary equation is $(r+3)^2 = 0$, then one solution to the differential equation is $y = e^{-3t}$, and it turns out that another solution is $y = te^{-3t}$.

We can (and probably should) check this!

In conclusion, the **general** solution to the differential equation

$$y'' + 6y' + 9y = 0$$

is $y = c_1e^{-3t} + c_2te^{-3t}$.

WEEK 4 EXAMPLE 3: Solve the initial value problem.

$$y'' + 6y' + 9y = 0, \qquad y(0) = 2, \qquad y'(0) = 1$$