Auxiliary equations with complex roots

If we're given the differential equation $ay'' + by' + cy = 0$, then its 'auxiliary equation' is

$$ar^2 + br + c = 0$$

What happens if the auxiliary equation has complex roots?

Key idea: There will *still* be solutions of the form e^{rt} even when r is a complex number!

WEEK 5 EXAMPLE 1: Find two independent solutions to the differential equation.

$$y'' - 6y' + 10y = 0$$

The auxiliary equation for Example 1 is

$$r^2 - 6r + 10 = 0$$

We can solve this with the quadratic formula or by completing the square.

The solutions to $r^2 - 6r + 10 = 0$ are $r = 3 + i$ and $r = 3 - i$.

So in fact, two solutions for Example 4 will be

$$y = e^{(3+i)t} \qquad \text{and} \qquad y = e^{(3-i)t}$$

How do we get solutions that don't involve complex numbers?

Fact: $e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t}$ just using rules of exponents.

More subtle fact:

$$e^{i\beta t} = \cos(\beta t) + i\sin(\beta t)$$

(The fact $e^{ix} = \cos x + i\sin x$ is sometimes called Euler's formula and can be shown by plugging into the power series for e^x.)

So the functions $e^{(\alpha+i\beta)t}$ and $e^{(\alpha-i\beta)t}$ can be rewritten in the following way:

$$e^{(\alpha+i\beta)t} = e^{\alpha t}e^{i\beta t} = e^{\alpha t}\big(\cos(\beta t) + i\sin(\beta t)\big)$$
$$e^{(\alpha-i\beta)t} = e^{\alpha t}e^{-i\beta t} = e^{\alpha t}\big(\cos(\beta t) - i\sin(\beta t)\big)$$

SUMMARY: If $\alpha \pm i\beta$ is a pair of complex solutions of the auxiliary equation, then two independent solutions of the differential equation are

$$e^{\alpha t}\cos(\beta t) \qquad \text{and} \qquad e^{\alpha t}\sin(\beta t)$$

Example 1 revisited

Differential equation was $y'' - 6y' + 10y = 0$

Auxiliary equation was $r^2 - 6r + 10 = 0$

Roots of auxiliary equation are $r = 3 \pm 1i$ (so $\alpha = 3$ and $\beta = 1$)

Two independent solutions of the differential equation are

$$y_1 = e^{3t} \cos t \qquad \text{and} \qquad y_2 = e^{3t} \sin t$$

so the general solution is $y = c_1 e^{3t} \cos t + c_2 e^{3t} \sin t$.

WEEK 5 EXAMPLE 2: Find the general solution of the differential equation.

$$y'' + 10y' + 41y = 0$$

WEEK 5 EXAMPLE 3: Solve the initial value problem.

$$y'' - 2y' + 2y = 0, \qquad y(\pi) = e^{\pi}, \qquad y'(\pi) = 0$$

Nonhomogeneous equations

So far, we have learned techniques for solving *homogeneous* linear differential equations with constant coefficients.

Here, 'homogeneous' means that the right-hand side is 0. For example:

$$y'' - 10y' + 26y = 0$$
$$2y'' + 13y' - 7y = 0$$
$$y''' - 4y'' + 7y' - 6y = 0$$

What do we do if the right-hand side is **not** 0? For example:

$$y'' - 2y' - 3y = 9t$$

How might we find a solution of this differential equation?

In essence, we *guess* the form of a solution! In this case, maybe another linear function of t.

We can try $y = At + B$ where A and B are *unknown* constants. Then:

$$y = At + B$$
$$y' = A$$
$$y'' = 0$$

We can then plug these into the nonhomogeneous equation that we're trying to solve.

After some algebra, we find that a particular solution of the nonhomogeneous differential equation

$$y'' - 2y' - 3y = 9t$$

is given by $y = -3t + 2$.

What about a different nonhomogeneous equation? For example:

$$y'' - 2y' - 3y = 10e^{4t}$$

Can we 'guess' a solution with undetermined coefficients?

If we try $y = Ae^{4t}$, then we have

$$y = Ae^{4t}$$
$$y' = 4Ae^{4t}$$
$$y'' = 16Ae^{4t}$$

Plug those into the nonhomogeneous equation we're trying to solve.

What about yet another nonhomogeneous equation? For example:

$$y'' - 2y' - 3y = -10\sin t$$

What form of particular solution should we guess?

Yet another nonhomogeneous differential equation:

$$y'' - 2y' - 3y = 8e^{3t}$$

What form of particular solution should we guess?

If we guess $y = Ae^{3t}$ and write down y' and y'', what happens?

If this doesn't work as a solution to the nonhomogeneous equation, how might we modify our guess?

The last differential equation on the previous page was

$$y'' - 2y' - 3y = 8e^{3t}$$

Since $y = e^{3t}$ is a solution to the corresponding *homogeneous* equation $y'' - 2y' - 3y = 0$, this means that $y = Ae^{3t}$ won't solve the nonhomogeneous equation.

But if we modify our guess to $y = Ate^{3t}$, it will turn out to work.

A more complicated example:

$$y'' - 10y' + 25y = te^{5t}$$

Here, $r = 5$ is a *double* root of the auxiliary equation $r^2 - 10r + 25 = 0$, so it turns out that we need to try a particular solution of the form

$$y = (At^3 + Bt^2 + Ct + D)e^{5t}$$

More nonhomogeneous equations, and what to try as particular solutions:

If given $y'' + 3y = -9$, we try $y = A$

If given $y'' + y = e^t$, we try $y = Ae^t$

If given $2y' + y = 3t^2$, we try $y = At^2 + Bt + C$

If given $y'' - y' + 9y = 3\sin 3t$, we try $y = A\sin 3t + B\cos 3t$

If given $y'' - 5y' + 6y = xe^x$, we try $y = (Ax + B)e^x$