The superposition principle

Consider the following two *nonhomogeneous* equations.

$$y'' + 9y = 18$$
$$y'' + 9y = 5e^t$$

If we want particular solutions of either of these, we would try $y = A$ for the first, and we would try $y = Ae^t$ for the second.

If we try those and do the algebra, it turns out that:

$y = 2$ is a particular solution of $y'' + 9y = 18$

$y = \frac{1}{2}e^t$ is a particular solution of $y'' + 9y = 5e^t$

What if we wanted *more* solutions of either of these nonhomogeneous equations? What if we wanted the *general* solution?

Also, what if we wanted a particular solution of this nonhomogeneous differential equation?

$$y'' + 9y = 18 + 5e^t$$

Would we need to start all over again, trying a solution of the form $y = A + Be^t$?

No! In fact, we can **superimpose** solutions.

Suppose $v(t)$ is a solution of $y'' + 9y = f(t)$.

Suppose $w(t)$ is a solution of $y'' + 9y = g(t)$.

It then turns out that $v(t) + w(t)$ is a solution of $y'' + 9y = f(t) + g(t)$.

(This works for any linear differential equations with the same left side, $y'' + 9y$ is just an example.)

Also, if we have a homogeneous linear differential equation

$$y'' + 9y = 0 \tag{1}$$

then any linear combination of solutions is a solution.

If $v(t)$ is a solution of (1), then $c \cdot v(t)$ is a solution.

If $v(t)$ and $w(t)$ are solutions of (1), then $v(t) + w(t)$ is a solution.

FACT: Given the *nonhomogeneous* differential equation

$$y'' + 9y = 18$$

its *general* solution is

$$y = 2 + c_1 \sin 3t + c_2 \cos 3t.$$

(This comes from adding a *particular* solution of the *nonhomogeneous* equation and the *general* solution of the corresponding *homogeneous* equation.)

Similarly, given the *nonhomogeneous* differential equation

$$y'' + 9y = 5e^t$$

its *general* solution is

$$y = \frac{1}{2}e^t + c_1 \sin 3t + c_2 \cos 3t.$$

Summary:

(general soln of nonhomog) = (particular soln of nonhomog) + (general soln of homog)

WEEK 6 EXAMPLE 1: Find the general solution of the differential equation.

$$y'' - 3y' + 2y = e^x \sin x$$

(Strategy: Find the general solution of the associated homogeneous equation, and also find a particular solution by guessing a solution of the right form.)

WEEK 6 EXAMPLE 2: Solve the initial value problem.

$$y'' + 9y = 27, \qquad y(0) = 4, \qquad y'(0) = 6$$

So far, we have seen

- a method to find *general* solutions to *homogeneous* equations
- guidelines for guessing *particular* solutions to *nonhomogeneous* equations.

Also, we have the guideline:

(general soln of nonhomog) = (particular soln of nonhomog) + (general soln of homog)

In Section 4.4 (undetermined coefficients) we saw that IF the right side of the nonhomogeneous equation is a product of polynomials, exponentials, sines, and cosines, then we can guess another function of the same type for our particular solution.

For example, for the equation $5y'' - 3y' + 2y = t^3 \cos 4t$, we try

$$y_p = (At^3 + Bt^2 + Ct + D)\cos 4t + (Et^3 + Ft^2 + Gt + H)\sin 4t.$$

But what if the right side of the nonhomogeneous equation involves tangents, secants, logarithms, or negative powers?

For example, how should we guess a particular solution for these?

$$y'' + y = \tan t$$

$$y'' + 4y = \tan 2t$$

$$y'' + y = \sec t$$

$$y'' - 2y' + y = t^{-1}e^t$$

Variation of parameters

A nonhomogeneous equation has a corresponding homogeneous equation.

If our equation is second order, we can find two independent solutions of the homogeneous equation, call them $y_1 = y_1(t)$ and $y_2 = y_2(t)$.

Trick: When looking for a particular solution of the nonhomogeneous equation, try $y_p = v_1(t)y_1(t) + v_2(t)y_2(t)$ where v_1 and v_2 are *functions*.

WEEK 6 EXAMPLE 3: Find a particular solution of the nonhomogeneous equation.

$$y'' + y = \sec t$$

First, note that the corresponding homogeneous equation is $y'' + y = 0$. We know that two independent solutions of that equation are $y_1 = \cos t$ and $y_2 = \sin t$.

We will try to find a particular solution of the form $y_p = v_1y_1 + v_2y_2$.

WEEK 6 EXAMPLE 4: Find a particular solution of the nonhomogeneous equation.

$$y'' - 2y' + y = t^{-1}e^t$$

The corresponding homogeneous equation is $y'' - 2y' + y = 0$, and the auxiliary equation is $r^2 - 2r + 1 = 0$ or $(r-1)^2 = 0$, so two independent solutions of the homogeneous equation are $y_1 = e^t$ and $y_2 = te^t$.

We'll look for a particular solution of the form $y_p = v_1y_1 + v_2y_2$.

In general, suppose we have a second order nonhomogeneous equation

$$ay'' + by' + cy = f(t)$$

and suppose y_1 and y_2 are two independent solutions of the corresponding homogeneous equation.

If we try a particular solution of the form $y_p = v_1 y_1 + v_2 y_2$ and impose the condition $v_1' y_1 + v_2' y_2 = 0$, then plugging y_p and its derivatives into the differential equation leads to the following:

If we can find two functions v_1 and v_2 that satisfy the system

$$v_1' y_1 + v_2' y_2 = 0$$

$$v_1' y_1' + v_2' y_2' = \frac{f(t)}{a}$$

then $y_p = v_1 y_1 + v_2 y_2$ will be a particular solution of the nonhomogeneous equation $ay'' + by' + cy = f(t)$.