Variable coefficient equations

The differential equations we have studied lately, whether homogeneous or nonhomogeneous, have all had *constant* coefficients on the left side.

But linear differential equations can also have functions of the independent variable as coefficients.

For example: $(t-3)y'' + y' + \sqrt{t}\,y = \ln t$

One important special case is a 'Cauchy-Euler' or 'equidimensional' equation, which has the form

$$at^2y''(t) + bty'(t) + cy(t) = f(t)$$

An example is $t^2y'' + 7ty' - 7y = 0$.

For these equations, guessing $y = t^r$ turns out to be helpful.

WEEK 7 EXAMPLE 1: By starting with the guess $y = t^r$, find the general solution of the homogeneous differential equation

$$t^2y'' + 7ty' - 7y = 0.$$

(If we try $y = t^r$, what are y' and y''? Why might this work?)

The guess $y = t^r$ led to an associated algebraic equation (which we could call the 'characteristic' equation).

What should we do if the characteristic equation has repeated roots or complex roots?

WEEK 7 EXAMPLE 2: Find two independent real-valued solutions of the differential equation.

$$t^2y'' + 7ty' + 13y = 0$$

FACT: If $r = \alpha \pm \beta i$ are two complex roots of the characteristic equation, then

$$y = t^\alpha \cos(\beta \ln t) \qquad \text{and} \qquad y = t^\alpha \sin(\beta \ln t)$$

are two independent solutions of the Cauchy-Euler equation.

FACT: If r is a repeated root of the characteristic equation, then

$$y = t^r \qquad \text{and} \qquad y = t^r \ln t$$

are two independent solutions of the Cauchy-Euler equation.

WEEK 7 EXAMPLE 3: Find two independent solutions of the differential equation.

$$t^2y'' - 9ty' + 25y = 0$$

Reduction of order

If we have a second-order differential equation and one solution $y_1(t)$, we might be able to find another (independent) solution of the form $y(t) = v(t)y_1(t)$.

WEEK 7 EXAMPLE 4: Find a solution of

$$ty'' - (t+1)y' + y = 0$$

given that $y_1(t) = e^t$ is a solution.

It is also possible to use the method of 'variation of parameters' for nonhomogeneous Cauchy-Euler equations.

Examples:

$$t^2y'' + ty' + 9y = -\tan(3\ln t)$$
$$t^2y'' + 3ty' + y = t^{-1}$$