The Laplace Transform

The Laplace transform is a tool that can turn differential equations into algebraic equations.

Definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

This transforms a function of t into a function of s.

Why would we do this?

We've already seen that we sometimes write functions as infinite series.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Another example is Fourier series, which are infinite series involving sine and cosine functions. For example:

$$\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots$$

To figure out the coefficients, we compute some *integrals*.

We can compute the Laplace transforms of some common functions.

EXAMPLE: Find the Laplace transform of $f(t) = 1$.

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt = \cdots = \frac{1}{s}$$

EXAMPLE: Find the Laplace transform of $f(t) = e^{17t}$.

$$\mathcal{L}\{e^{17t}\} = \int_0^\infty e^{-st} \cdot e^{17t} \, dt = \cdots = \frac{1}{s-17}$$

We can find the Laplace transforms of other common functions such as $f(t) = t$, t^2, t^3, $\ldots$, $\sin t$, $\cos t$, $\sin 17t$, $\cos 17t$, $\ldots$

The method for doing this is integration by parts.

To save time, we could remember the Laplace transforms of some common functions, or keep track of them in a table.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1!}{s^2}$
t^2	$\frac{2!}{s^3}$
t^3	$\frac{3!}{s^4}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin t$	$\frac{1}{s^2+1}$
$\cos t$	$\frac{s}{s^2+1}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$

Properties of the Laplace transform

One key property is **linearity**.

$$\mathcal{L}\{c \cdot f(t)\} = c \cdot \mathcal{L}\{f(t)\}$$
$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

These properties just come from the fact that integration is linear.

Why are Laplace transforms useful in differential equations?

Something nice happens with the Laplace transform of a *derivative*.

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt = \cdots$$

(We need a technical condition here: $f(t)$ must not grow too quickly.)

It turns out that we have

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0) \\ \mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ \mathcal{L}\{f'''(t)\} &= s^3\mathcal{L}\{f(t)\} - s^2 f(0) - sf'(0) - f''(0) \end{aligned}$$

and so on.

(Rough idea: Differentiating with respect to t is like multiplying by s, except we also need to worry about initial values.)

An example to play with: How would we solve this initial value problem either *with* or *without* Laplace transforms?

$$y'' + y = 2 + t^2, \qquad y(0) = 0, \qquad y'(0) = 0$$

Another example: How would we solve this initial value problem (with or without Laplace transforms)?

$$y'' + 2y' + y = 6\sin t - 4\cos t, \qquad y(0) = -1, \qquad y'(0) = 1$$

Another fact about Laplace transforms: If $\mathcal{L}\{f(t)\} = F(s)$, then

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

That is, multiplying $f(t)$ by e^{at} corresponds to replacing s with $s-a$.

We can use this to expand our table of Laplace transforms (and inverse Laplace transforms).

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1!}{s^2}$
t^2	$\frac{2!}{s^3}$
t^3	$\frac{3!}{s^4}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin t$	$\frac{1}{s^2+1}$
$\cos t$	$\frac{s}{s^2+1}$
$\sin bt$	$\frac{b}{s^2+b^2}$
$\cos bt$	$\frac{s}{s^2+b^2}$
e^{at}	$\frac{1}{s-a}$
$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$

When we calculate Laplace transforms, we might get complicated rational functions of s.

We may be able to rewrite those as sums of simpler functions using the method of partial fractions.

Find the inverse Laplace transforms of the given functions.

$$F(s) = \frac{6}{(s-1)^4}$$

$$F(s) = \frac{s+1}{s^2+2s+10}$$

$$F(s) = \frac{1}{s^2+4s+8}$$

Determine the partial fraction expansions of the given functions. Can you use that to find the inverse Laplace transforms?

$$F(s) = \frac{s^2 - 26s - 47}{(s-1)(s+2)(s+5)}$$

$$F(s) = \frac{-s-7}{(s+1)(s-2)}$$

$$F(s) = \frac{-2s^2 - 3s - 2}{s(s+1)^2}$$