Infinite limits

When we say 'infinite limits', we mean that the **output** of the function approaches $\pm\infty$.

One situation where this happens is fractions where direct substitution gives nonzero/zero.

FACT: We have

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to 0} \frac{1}{x} \text{ DNE}$$

QUESTION: Evaluate the following.

$$\lim_{x \to 3^-} \frac{1}{x-3} \qquad \lim_{x \to 3^+} \frac{1}{x-3} \qquad \lim_{x \to 3} \frac{1}{x-3}$$

QUESTION: Evaluate the following.

- $\displaystyle\lim_{x \to 5^+} \frac{-8}{x-5}$

- $\displaystyle\lim_{x \to 4^-} \frac{3}{4-x}$

- $\displaystyle\lim_{x \to -3^-} \frac{4x}{x+3}$

Vertical asymptotes

If either $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ are ∞ or $-\infty$, then f has a **vertical asymptote** at $x = c$.

QUESTION: Find all vertical asymptotes of the function.

$$f(x) = \frac{x^2 + x - 2}{x^2 - 4x + 3}$$

Limits at infinity

Limits 'at' infinity mean that the **input** of the function approaches $\pm\infty$.

FACT: We have

$$\lim_{x\to-\infty}\frac{1}{x}=0 \qquad\qquad \lim_{x\to\infty}\frac{1}{x}=0$$

In fact, any limit of the form 'finite/infinite' will approach 0.

For example, $\displaystyle\lim_{x\to-\infty}\frac{1}{x^n}$ and $\displaystyle\lim_{x\to\infty}\frac{1}{x^n}$ are both equal to 0 for any positive n.

Horizontal asymptotes

If $\displaystyle\lim_{x\to-\infty} f(x)=L$ or $\displaystyle\lim_{x\to\infty} f(x)=L$ (or both) then f has a **horizontal asymptote** at $y=L$.

Limits at infinity of powers and polynomials

- If n is even, then $\displaystyle\lim_{x\to-\infty} x^n=\infty$ and $\displaystyle\lim_{x\to\infty} x^n=\infty$
- If n is odd, then $\displaystyle\lim_{x\to-\infty} x^n=-\infty$ and $\displaystyle\lim_{x\to\infty} x^n=\infty$

Also, if $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ is any polynomial, then

$$\lim_{x\to\pm\infty} p(x)=\lim_{x\to\pm\infty} a_nx^n$$

That is, the limits at infinity of a polynomial are the same as that of its dominant term.

EXAMPLE: Find the limits when $x\to\pm\infty$ of $f(x)=x^2-3x+2$ and $g(x)=x^3-4x$.

Limits at infinity for rational functions

For these, we can divide top and bottom by the largest power appearing in the denominator.

EXAMPLE: Evaluate the limit.

$$\lim_{x\to\infty}\frac{x+x^3-8x^4}{2x^4+x^2-1}$$

EXAMPLE: Evaluate the limit.

$$\lim_{x\to-\infty}\frac{x^5+4x^2-5}{x^3+6x}$$