Average and instantaneous rate of change

Recall that the *average rate of change* of the function f on the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}$$

This is the slope of the 'secant line' joining the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ on the graph.

If h is a small number, the average rate of change of f on the interval $[a, a + h]$ is

$$\frac{f(a + h) - f(a)}{h}$$

We then define the **instantaneous** rate of change of f **at** the point $x = a$ to be

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

This is the **slope of the tangent line** to the graph of f **at** the point $\big(a, f(a)\big)$.

If we have a **point** on a graph, and we know the **slope** of the tangent line at that point, then we can write an **equation** of the tangent line at that point.

If the point is $\big(a, f(a)\big)$, and the slope is m, then an equation of the line is

$$y - f(a) = m(x - a)$$

EXAMPLE: Consider the function $f(x) = x^2$ at the point where $x = 3$.

(i) Find the **slope** of the tangent line at that point.

(ii) Find an **equation** of the tangent line at that point.

The derivative function

As on the previous page, the slope of the tangent line to the graph of f at $x = a$ is

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

where a can be any number. So we could also say that

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is the slope of the tangent line to the graph of f at a general x.

This is the definition of the **derivative** of f. (It's like a 'slope-predictor' function for f.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative of f is a **rate** of change (change in output divided by change in input). So we sometimes call this the derivative of f 'with respect to' x, and sometimes write it as

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

We also write $\frac{d}{dx}$ to mean 'the derivative of'.

EXAMPLE: Use the definition of the derivative to find the derivative of $f(x) = \frac{1}{x}$.

EXAMPLE: Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$.

Tangent lines and linear approximation

Consider the graph of f at $x = a$. The **slope** of the tangent line at that point is $f'(a)$.

An **equation** of the tangent line at that point is

$$y - f(a) = f'(a) \cdot (x - a) \tag{1}$$

EXAMPLE: Find an equation of the tangent line to $f(x) = \dfrac{1}{x}$ at $x = 10$.

The equation of the tangent line above (1) can be rearranged slightly as

$$y = f(a) + f'(a) \cdot (x - a)$$

The function $L(x) = f(a) + f'(a) \cdot (x - a)$ is called the **linear approximation** of f at a.

EXAMPLE:
Find the linear approximation of $f(x) = \sqrt{x}$ at $a = 25$. Then use this to estimate $\sqrt{23}$.

Differentiability

If the derivative of f exists at a point, then f is called 'differentiable' at that point.

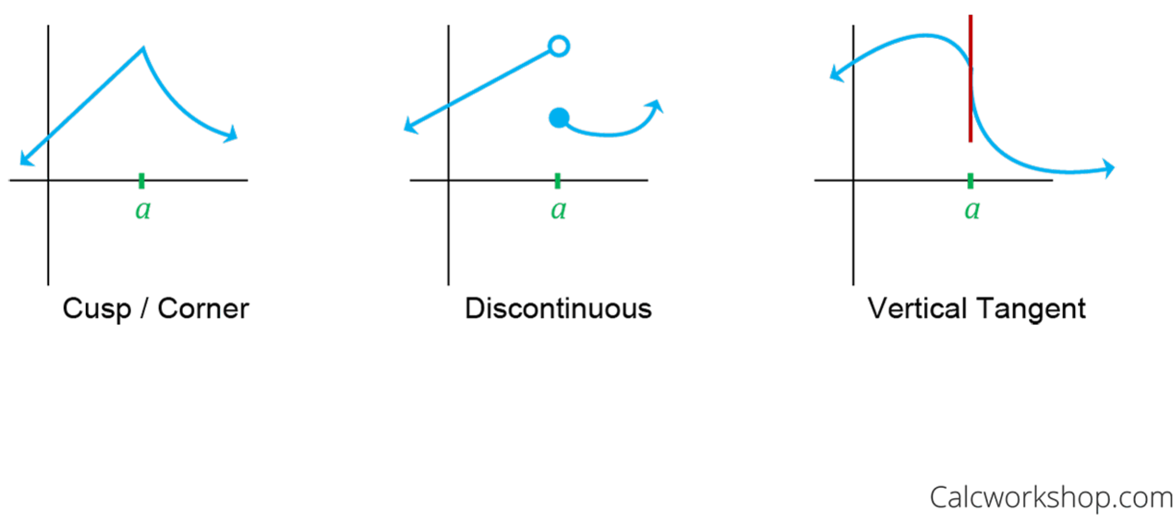
QUESTION: Where is $f(x) = \sqrt{x}$ differentiable, and where is it not differentiable?

Fact: Differentiability implies continuity, but not the other way around.

Informally, 'continuous' means you can draw the graph without lifting your pen.

'Differentiable' means the slope at a point has a well-defined finite value.

Below, you can see various ways that a function can fail to be differentiable at a point.



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