MAC 2312

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Miscellaneous series practice

You do not need to hand this in.

Question 1. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{(n+2)(n+3)}$$

n^{th} term test. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)} = \lim_{n\to\infty} \frac{n^2+n}{n^2+5n+6}$

$= \lim_{n\to\infty} \frac{n^2+n}{n^2+5n+6} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{5}{n}+\frac{6}{n^2}} = \frac{1+0}{1+0+0} = 1$

Since $\lim_{n\to\infty} a_n$ is not 0, the series $\sum a_n$ diverges.

(Note: You are allowed and encouraged to notice and use shortcuts in the above algebra. We have the ratio of two degree 2 polynomials.)

Question 2. Determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1}{n+4}$$

Series is $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots$

This is just the harmonic series with the first few terms removed. Removing finitely many terms doesn't affect convergence or divergence. DIVERGES

(Integral test will also work)

Question 3. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+3}$$

Limit comparison test. $a_n = \frac{n}{n^2+3}$. Choose $b_n = \frac{n}{n^2} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{n}{n^2+3} \div \frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{n}{n^2+3}\cdot\frac{n}{1}\right)$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2+3} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{n^2}} = \frac{1}{1+0} = 1$$

Since $\lim_{n\to\infty} \frac{a_n}{b_n}$ is not 0 and not ∞, $\sum a_n$ and $\sum b_n$ "behave the same"

We know $\sum b_n = \sum \frac{1}{n}$ diverges, so $\sum a_n = \sum \frac{n}{n^2+3}$ DIVERGES

Question 4. Determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} e^{-2n}$$

$$\text{Series} = \sum \frac{1}{e^{2n}} = \sum \left(\frac{1}{e^2}\right)^n$$

This is a geometric series with $r = \frac{1}{e^2}$.

Note $2<e<3$ so $4<e^2<9$ so $\frac{1}{9} < \frac{1}{e^2} < \frac{1}{4}$

So $-1<r<1$ so this geometric series <u>converges</u>.

Question 5. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2}{10^n}$$

Series $= 2 \cdot \sum \frac{1}{10^n} = 2 \cdot \sum \left(\frac{1}{10}\right)^n$ or $\sum 2 \cdot \left(\frac{1}{10}\right)^n$

Geometric series with $r = \frac{1}{10}$

Since $-1 < r < 1$, the series CONVERGES

Question 6. Determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{n!}{1000^n}$$

Ratio test: $a_n = \dfrac{n!}{1000^n}$ $\quad a_{n+1} = \dfrac{(n+1)!}{1000^{n+1}}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)!}{1000^{n+1}} \div \frac{n!}{1000^n} \right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)!}{1000^{n+1}} \cdot \frac{1000^n}{n!} \right) = \lim_{n\to\infty} \left(\frac{(n+1)!}{n!} \cdot \frac{1000^n}{1000^{n+1}} \right)$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{1} \cdot \frac{1}{1000} \right) = \lim_{n\to\infty} \frac{n+1}{1000} = \infty \text{ which is } > 1$$

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, the series DIVERGES

Question 7. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n + 4^n}{3^n + 4^n}$$

n^{th} term test: $\displaystyle \lim_{n\to\infty} \frac{2^n + 4^n}{3^n + 4^n} \cdot \frac{\frac{1}{4^n}}{\frac{1}{4^n}} = \lim_{n\to\infty} \frac{\frac{2^n}{4^n} + \frac{4^n}{4^n}}{\frac{3^n}{4^n} + \frac{4^n}{4^n}}$

$\displaystyle = \lim_{n\to\infty} \frac{\left(\frac{2}{4}\right)^n + 1}{\left(\frac{3}{4}\right)^n + 1} = \frac{0+1}{0+1} = \frac{1}{1} = 1$

Since the n^{th} term does not approach 0,
the series diverges.

Question 8. Determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

Fact: We know $\ln n > 1$ if $n \geq 3 > e$

Therefore $\displaystyle \frac{\ln n}{n} > \frac{1}{n}$

Therefore $\displaystyle \sum \frac{\ln n}{n} > \sum \frac{1}{n}$

Therefore $\displaystyle \sum \frac{\ln n}{n}$ diverges by direct comparison with harmonic series

(Integral test will also work. The integral involves u-substitution.)

Question 9. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2}{e^n}$$

Ratio test. $a_n = \dfrac{n^2}{e^n}$ $\quad a_{n+1} = \dfrac{(n+1)^2}{e^{n+1}} = \dfrac{n^2+2n+1}{e^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n^2+2n+1}{e^{n+1}} \div \frac{n^2}{e^n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n^2+2n+1}{e^{n+1}} \cdot \frac{e^n}{n^2}\right) = \lim_{n\to\infty}\left(\frac{n^2+2n+1}{n^2} \cdot \frac{e^n}{e^{n+1}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}+\frac{1}{n^2}}{1} \cdot \frac{1}{e}\right) = \frac{1}{e} < 1.$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ the series CONVERGES

Question 10. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{7}{\sqrt{n+4}}$$

Limit comparison test. $a_n = \dfrac{7}{\sqrt{n+4}}$ Choose $b_n = \dfrac{1}{\sqrt{n}}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{7}{\sqrt{n+4}} \div \frac{1}{\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{7}{\sqrt{n+4}} \cdot \frac{\sqrt{n}}{1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{7}{1} \cdot \frac{\sqrt{n}}{\sqrt{n+4}}\right) = \lim_{n\to\infty}\left(\frac{7}{1} \cdot \sqrt{\frac{n}{n+4}}\right) = \lim_{n\to\infty}\left(\frac{7}{1} \cdot \sqrt{\frac{1}{1+\frac{4}{n}}}\right)$$

$$= \frac{7}{1} \cdot \sqrt{\frac{1}{1+0}} = 7$$ which is not 0 and not ∞.

Therefore $\sum a_n$ behaves like $\sum b_n = \sum \frac{1}{n^{1/2}}$ which is a p-series with $p = \frac{1}{2} \leq 1$. DIVERGES

Question 11. Determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{5n + 10\sqrt{n}}$$

Limit comparison test. $a_n = \dfrac{1}{5n+10n^{1/2}}$ Choose $b_n = \dfrac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty}\left(\frac{1}{5n+10n^{1/2}} \div \frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{1}{5n+10n^{1/2}} \cdot \frac{n}{1}\right)$$

$$= \lim_{n\to\infty} \frac{n}{5n+10n^{1/2}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{5+\frac{10}{n^{1/2}}} = \frac{1}{5+0} = \frac{1}{5}$$

Since $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ is not 0 and not ∞, we conclude that $\sum a_n$ behaves the "same" as $\sum b_n = \sum \frac{1}{n}$, which DIVERGES

Question 12. Determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$$

We know $\ln n > 1$ if $n \geq 3$

Therefore $\dfrac{\ln n}{\sqrt{n}} > \dfrac{1}{\sqrt{n}}$

Therefore $\sum \dfrac{\ln n}{\sqrt{n}} > \sum \dfrac{1}{\sqrt{n}} = \sum \dfrac{1}{n^{1/2}}$ which is a p-series with $p = \frac{1}{2} \leq 1$ DIVERGES

Therefore $\sum \dfrac{\ln n}{\sqrt{n}}$ diverges by basic comparison

(Can also use limit comparison test with $b_n = \frac{5^n}{4^n} = \left(\frac{5}{4}\right)^n$)

Question 13. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$$

n^{th} term test. $\lim_{n\to\infty} \frac{5^n}{4^n+3} \cdot \frac{\frac{1}{5^n}}{\frac{1}{5^n}}$

$$= \lim_{n\to\infty} \frac{1}{\frac{4^n}{5^n}+\frac{3}{5^n}} = \lim_{n\to\infty} \frac{1}{\left(\frac{4}{5}\right)^n+\frac{3}{5^n}}$$

$= \frac{1}{0+0} = \infty$, not 0. Since n^{th} term doesn't approach 0, the series diverges.

Question 14. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n+1}$$

n^{th} term test. $\lim_{n\to\infty} \frac{2^n}{n+1} = \lim_{x\to\infty} \frac{2^x}{x+1}$ $\quad \frac{\infty}{\infty}$ form

Can use L'Hopital's rule. $f(x) = 2^x \Rightarrow f'(x) = 2^x \cdot \ln 2$

$g(x) = x+1 \Rightarrow g'(x) = 1$

$$\lim_{x\to\infty} \frac{2^x}{x+1} = \lim_{x\to\infty} \frac{2^x \cdot \ln 2}{1} = \infty, \text{ not } 0.$$

Since n^{th} term doesn't approach 0, the series diverges.

Question 15. Determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{\sqrt{n}}{\ln n}$$

n^{th} term test. $\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{x\to\infty} \frac{\sqrt{x}}{\ln x}$ $\frac{\infty}{\infty}$ form

L'Hopital. $f(x) = \sqrt{x} = x^{1/2} \Rightarrow f'(x) = \frac{1}{2}x^{-1/2}$

$g(x) = \ln x \Rightarrow g'(x) = \frac{1}{x}$

$$\lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} = \lim_{x\to\infty} \frac{\frac{1}{2}x^{-1/2}}{\frac{1}{x}} = \lim_{x\to\infty}\left(\frac{1}{2}\cdot\frac{1}{x^{1/2}} \div \frac{1}{x}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{2}\cdot\frac{1}{x^{1/2}}\cdot\frac{x}{1}\right) = \lim_{x\to\infty}\frac{x^{1/2}}{2} = \infty$$

not 0

Question 16. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$$

We know $-1 \le \cos n \le 1$. So $0 \le \cos^2 n \le 1$

Therefore $\frac{\cos^2 n}{n^{3/2}} \le \frac{1}{n^{3/2}}$

$$\sum \frac{\cos^2 n}{n^{3/2}} \le \sum \frac{1}{n^{3/2}}$$

The given series converges by direct comparison with $\sum \frac{1}{n^{3/2}}$ which is a convergent p-series $(p = \frac{3}{2} > 1)$

Question 17. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}$$

Direct comparison: $n \cdot 3^n \geq 3^n \Rightarrow \frac{1}{n \cdot 3^n} \leq \frac{1}{3^n}$

$\sum \frac{1}{n \cdot 3^n} \leq \sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$ which is a convergent geometric series with $r = \frac{1}{3}$ $(-1 < r < 1)$

So the given series <u>converges</u> by direct comparison.

Can also use ratio test. $a_n = \frac{1}{n \cdot 3^n}$ $a_{n+1} = \frac{1}{(n+1) \cdot 3^{n+1}}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n} = \frac{1}{(n+1)3^{n+1}} \cdot \frac{n \cdot 3^n}{1} = \frac{n}{n+1} \cdot \frac{1}{3} \to \frac{1}{3} < 1$

Question 18. Determine whether the series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$$

We know $\ln n$ grows more slowly than any power of n.

For example, $\ln n < \sqrt{n}$. Then $\frac{\ln n}{n^2} < \frac{\sqrt{n}}{n^2}$

$\frac{\ln n}{n^2} < \frac{n^{1/2}}{n^2} = \frac{1}{n^{3/2}} \Rightarrow \sum \frac{\ln n}{n^2} \leq \sum \frac{1}{n^{3/2}}$

So the given series CONVERGES by direct comparison with a p-series with $p = \frac{3}{2} > 1$

Question 19. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{n^2}$$

Ratio test. $a_n = \frac{n!}{n^2}$, $a_{n+1} = \frac{(n+1)!}{(n+1)^2} = \frac{(n+1)!}{n^2+2n+1}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{(n+1)!}{n^2+2n+1} \cdot \frac{n^2}{n!} \right)$$

$$= \lim_{n\to\infty} \left(\frac{(n+1)!}{n!} \cdot \frac{n^2}{n^2+2n+1} \right) = \lim_{n\to\infty} \left(\underbrace{\frac{n+1}{1}}_{\to\infty} \cdot \underbrace{\frac{1}{1+\frac{2}{n}+\frac{1}{n^2}}}_{\to 1} \right)$$

$= \infty > 1$, so by the Ratio test, the given series DIVERGES.

Question 20. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n3^n}$$

METHOD 1. Basic comparison. $n \geq 1$ so $n \cdot 3^n \geq 3^n$

so $\frac{2^n}{n \cdot 3^n} \leq \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n$. So $\sum \frac{2^n}{n \cdot 3^n} \leq \sum \left(\frac{2}{3}\right)^n$

So the given series CONVERGES by direct comparison with a geometric series with $r = \frac{2}{3}$.

METHOD 2. Ratio test will work. $a_n = \frac{2^n}{n \cdot 3^n}$, $a_{n+1} = \frac{2^{n+1}}{(n+1) \cdot 3^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1) \cdot 3^{n+1}} \cdot \frac{n \cdot 3^n}{2^n} = \frac{2}{3} \cdot \frac{n}{n+1} \to \frac{2}{3} < 1$$

Question 21. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^{1.4}}{2^n}$$

Ratio test. $a_n = \dfrac{n^{1.4}}{2^n}$ $\quad a_{n+1} = \dfrac{(n+1)^{1.4}}{2^{n+1}}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{a_{n+1}}{a_n} = \frac{(n+1)^{1.4}}{2^{n+1}} \cdot \frac{2^n}{n^{1.4}} = \left(\frac{n+1}{n}\right)^{1.4} \cdot \frac{1}{2}$$

so $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{1.4} \cdot \frac{1}{2} = 1^{1.4} \cdot \frac{1}{2} = \frac{1}{2} < 1$

So by ratio test, the given series CONVERGES

Question 22. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^{10}}{10^n}$$

Ratio test. $a_n = \dfrac{n^{10}}{10^n}$ $\quad a_{n+1} = \dfrac{(n+1)^{10}}{10^{n+1}}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{(n+1)^{10}}{10^{n+1}} \cdot \frac{10^n}{n^{10}}\right)$$

$$= \lim_{n\to\infty}\left(\underbrace{\left(\frac{n+1}{n}\right)}_{\to 1}^{10} \cdot \frac{1}{10}\right) = 1^{10} \cdot \frac{1}{10} = 1 \cdot \frac{1}{10} = \frac{1}{10} < 1$$

So by the ratio test, the given series CONVERGES.

Question 23. Determine whether the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{1.25^n}$$

We know $(-1)^n = 1$ or -1, so $2+(-1)^n$ is either $2+1$ or $2-1$. That is, $1 \leq 2+(-1)^n \leq 3$

Therefore $\frac{2+(-1)^n}{1.25^n} \leq \frac{3}{1.25^n}$

so $\sum \frac{2+(-1)^n}{1.25^n} \leq \sum \frac{3}{1.25^n} = \sum 3 \cdot \left(\frac{1}{1.25}\right)^n$

which is a geometric series with $r = \frac{1}{1.25} < 1$.

Given series CONVERGES using direct comparison

Question 24. Determine whether the series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$$

Ratio test. $a_n = \frac{(n!)^2}{(2n)!}$ $a_{n+1} = \frac{((n+1)!)^2}{(2(n+1))!} = \frac{((n+1)!)^2}{(2n+2)!}$

$$\frac{a_{n+1}}{a_n} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \left(\frac{n+1}{1}\right)^2 \cdot \frac{1}{(2n+1)(2n+2)} = \frac{n^2+2n+1}{4n^2+6n+2} \to \frac{1}{4} < 1$$

Series CONVERGES

Question 25. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$$

Series is alternating. Series converges "as is" by alternating series test since "positive part" $\frac{1}{\sqrt{n}}$ goes to 0.

But if we replace each term with its absolute value, we get the series $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ which diverges (p-series with $p = \frac{1}{2} \leq 1$).

So the original series CONVERGES CONDITIONALLY.

Question 26. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{3/2}}$$

If we replace each term with its absolute value, we get the series $\sum \frac{1}{n^{3/2}}$ which converges (p-series with $p = \frac{3}{2} > 1$).

So the original series CONVERGES ABSOLUTELY.

Question 27. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n3^n}$$

$\sum |a_n| = \sum \frac{1}{n3^n} \leq \sum \frac{1}{3^n}$ by direct comparison (since $n3^n \geq 3^n$)

Note $\sum \frac{1}{3^n} = \sum \left(\frac{1}{3}\right)^n$ which is a convergent geometric series ($r = \frac{1}{3}$ satisfies $-1 < r < 1$)

Therefore the original series converges absolutely.

Question 28. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=2}^{\infty}(-1)^n\frac{4}{(\ln n)^2}$$

This is an alternating series where the positive part of the n^{th} term is $\frac{4}{(\ln n)^2}$, which decreases to 0, so the series converges "as is" by the alternating series test.

If we replace each term with its absolute value, we get the series $\sum \frac{4}{(\ln n)^2}$. We know $\ln n < \sqrt{n}$

so $\frac{4}{(\ln n)^2} > \frac{4}{(\sqrt{n})^2} = \frac{4}{n}$. So $\sum |a_n|$ diverges by direct comparison with a constant times the harmonic series

Original series CONVERGES CONDITIONALLY.

Question 29. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$$

This is an alternating series where the positive part of the n^{th} term is $\frac{n}{n^2+1}$, which decreases to 0, so the series converges "as is" by the alternating series test. If we replace each term with its absolute value, we get $\sum \frac{n}{n^2+1}$ which we can compare to $\sum \frac{n}{n^2} = \sum \frac{1}{n}$ using limit comparison. Original series CONVERGES CONDITIONALLY.

Question 30. Determine whether the series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2+5}{n^2+4}$$

n^{th} term test: We know $\lim\limits_{n\to\infty} \frac{n^2+5}{n^2+4} = 1$

(longer way: $\frac{n^2+5}{n^2+4} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \frac{1+\frac{5}{n^2}}{1+\frac{4}{n^2}} \to \frac{1+0}{1+0} = \frac{1}{1} = 1$)

So $(-1)^{n+1} \cdot \frac{n^2+5}{n^2+4}$ does <u>not</u> approach 0.

Series DIVERGES