

# Infinite Limits & Limits at Infinity

MAC 2311

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## 1 Infinite Limits

An **infinite limit** occurs when as the independent variable approaches a value, the dependent variable becomes arbitrarily large in magnitude. So, the function values increase or decrease without bound near a point.

Infinite limits are denoted by:

$$\lim_{x \rightarrow a} f(x) = \infty \text{ or } \lim_{x \rightarrow a} f(x) = -\infty$$

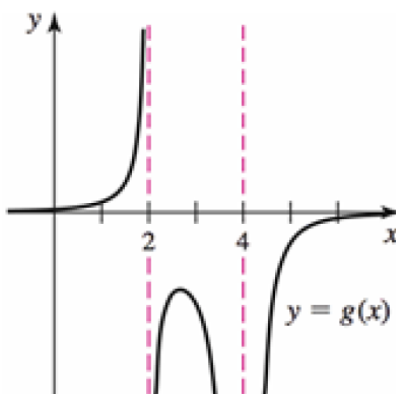
In the case of one-sided infinite limits:

$$\lim_{x \rightarrow a^-} f(x) = \infty \text{ or } \lim_{x \rightarrow a^-} f(x) = -\infty$$

$$\lim_{x \rightarrow a^+} f(x) = \infty \text{ or } \lim_{x \rightarrow a^+} f(x) = -\infty$$

*Note: In the event of an infinite limit, the limit does not exist since as  $x$  approaches some number, the function values become increasingly large in magnitude, never approaching a single, unique value. However, we don't use the notation of "DNE" to express this since we have a formal way to describe this occurrence.*

**Example 1:** Evaluate the following limits using the graph of  $g(x)$  given below.



- $\lim_{x \rightarrow 2^-} g(x)$

- $\lim_{x \rightarrow 2^+} g(x)$

- $\lim_{x \rightarrow 2} g(x)$

- $\lim_{x \rightarrow 4^-} g(x)$

- $\lim_{x \rightarrow 4^+} g(x)$

- $\lim_{x \rightarrow 4} g(x)$

## 1.1 Finding Infinite Limits Analytically

Many infinite limits are analyzed using a common arithmetic property:

The fraction  $\frac{a}{b}$  grows arbitrarily large in magnitude if the denominator,  $b$ , approaches 0 while the numerator,  $a$ , remains nonzero and relatively constant. In other words,  $\lim_{b \rightarrow 0} \frac{a}{b} = \pm\infty$ .

Let's take the following limits for example:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} \quad \lim_{x \rightarrow 0^-} \frac{1}{x} \quad \lim_{x \rightarrow 0} \frac{1}{x}$$

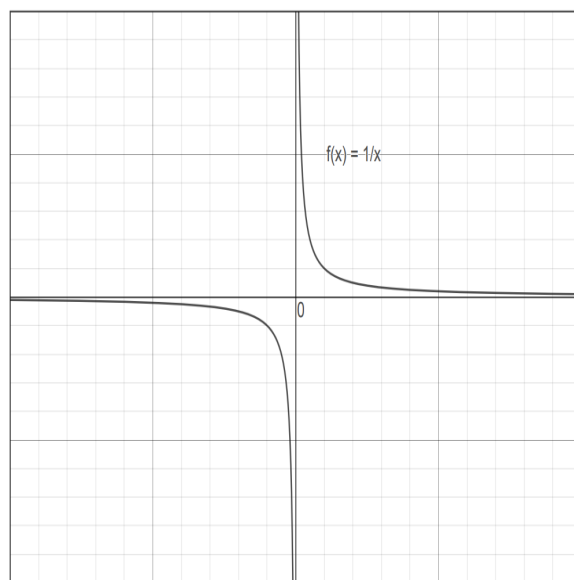
In all three cases, direct substitution does not work because we would be dividing a nonzero number by zero. So, let's observe what happens as we let  $x$  approach zero from either side. In the tables below, we can approach this numerically and by using the graph of  $f(x) = \frac{1}{x}$ , we can observe what happens graphically.

**$x$  approaches 0 from the right:**

$x$	1	0.1	0.01	0.001	0.0001	0.00001
$\frac{1}{x}$	1	10	100	1,000	10,000	100,000

**$x$  approaches 0 from the left:**

$x$	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001
$\frac{1}{x}$	-1	-10	-100	-1,000	-10,000	-100,000



We see, as  $x$  approaches zero from the right and left, the function values grow larger in magnitude. Specifically, as  $x$  approaches zero from the right, the function values grow more and more positive, while as  $x$  approaches zero from the left, the function values grow more and more negative. So, we can draw the following conclusions for each limit:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \quad \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty \quad \lim_{x \rightarrow 0} \frac{1}{x} \text{ DNE}$$

Let's look at the left-hand limit  $\left(\lim_{x \rightarrow 0^-} \frac{1}{x}\right)$  and solve it **analytically** (by figuring out how the function behaves without actually plugging in several values or without having to look at its graph).

- Since  $x$  is approaching zero from the left, this means that the  $x$ -values will be less than zero. So, they will be negative.
- In the numerator, what  $x$  is approaching does not matter since 1 is a constant. [*Recall the first of our common limits in techniques for computing limits.*]
- For the denominator,  $x$  is getting closer to zero and is negative.
- So, overall, we have a positive constant being divided by an increasingly small negative number. The result of this fraction will be an increasingly large negative number.

This gives us the following:

$$\lim_{x \rightarrow 0^-} \frac{\overbrace{1}^{\text{approaches 1}}}{\underbrace{x}_{\text{approaches 0, negative}}} = -\infty$$

**Example 2:** Evaluate each of the following limits:

$$\bullet \lim_{x \rightarrow 1^+} \frac{\overbrace{-2}^{\rightarrow -2}}{\underbrace{x-1}_{\rightarrow 0, +}} = -\infty$$

Since we are approaching 1 from the right, this means that  $x > 1$ . This implies  $x - 1 > 0$ . So,  $x - 1$  will get closer to zero and will be positive as  $x \rightarrow 1^+$ . Therefore, we have a negative constant being divided by an increasingly small positive number. The result will be a number which gets larger in magnitude and negative. So, the limit will be  $-\infty$ .

$$\bullet \lim_{x \rightarrow 1^-} \frac{-2}{x-1}$$

$$\bullet \lim_{x \rightarrow 2^-} \frac{x+1}{(x-2)^2}$$

## 1.2 Vertical Asymptotes

For any given function  $f(x)$  if any of the following limits are true, then  $f(x)$  will have a **vertical asymptote** at  $x = a$ .

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty \quad \lim_{x \rightarrow a^+} f(x) = \pm\infty \quad \lim_{x \rightarrow a} f(x) = \pm\infty$$

Only one of the above limits has to occur in order for a function to have a vertical asymptote at  $x = a$ .

**Example 3:** Let's revisit the function  $f(x) = \frac{1}{x}$ .

In previous work, we found the following:

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

From this we are able to draw the conclusion that the function  $f(x) = \frac{1}{x}$  has a vertical asymptote at  $x = 0$ .

### Finding Vertical Asymptotes of Rational Functions

Given a rational function,  $\frac{p(x)}{q(x)}$ , we can determine the vertical asymptotes, if any, with the following steps.

- Find the  $x$ -values where the denominator is zero, but the numerator is NOT zero.
  - Find  $x = a$  such that  $q(a) = 0$ , but  $p(a) \neq 0$
- Using the value(s) identified in the previous step, take one-sided limits of the function.
  - Find  $\lim_{x \rightarrow a^+} \frac{p(x)}{q(x)}$  and/or  $\lim_{x \rightarrow a^-} \frac{p(x)}{q(x)}$
- If either of the above limits go to  $\pm\infty$ , then your function has a vertical asymptote at  $x = a$ .

**Example 4:** Identify any vertical asymptotes of  $f(x) = \frac{x^2 + x - 2}{x^2 - 4x + 3}$

## 2 Limits at Infinity

A **limit at infinity** occurs when the independent variable increases or decreases without bound. So, limits at infinity tell us how a function is behaving as its  $x$ -values get increasingly more positive or negative.

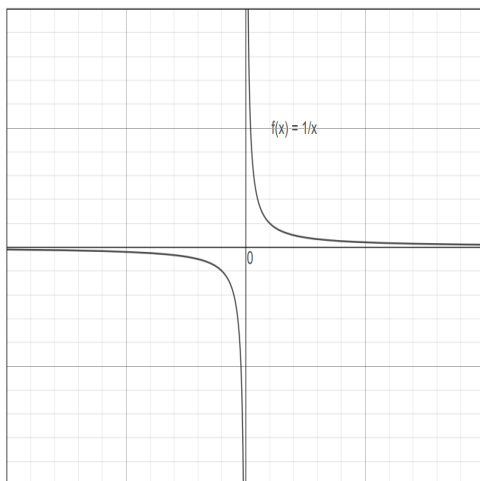
Limits at infinity are denoted by

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L$$

**Example 5:** Evaluate  $\lim_{x \rightarrow \infty} \frac{1}{x}$  and  $\lim_{x \rightarrow -\infty} \frac{1}{x}$

We note that direct substitution is not a valid approach for evaluating these limits since we can't simply plug in the values of  $\pm\infty$ . So, there are a couple of other approaches we can take.

- **Graphically:** Let's look at the graph of  $f(x) = \frac{1}{x}$  and observe what happens as the  $x$ -values grow larger in magnitude.



From the graph, we see that as  $x$  increases and decreases more and more, the function values get smaller and smaller in magnitude, approaching the value of zero. So, we are able to conclude:

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0 \text{ and } \lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

- **Analytically:** Similar to analyzing infinite limits, we can analyze limits at infinity using an arithmetic property:

The fraction  $\frac{a}{b}$  grows arbitrarily small in magnitude if the denominator,  $b$ , grows arbitrarily large in magnitude while the numerator,  $a$ , remains nonzero and relatively constant. In other words,  $\lim_{b \rightarrow \pm\infty} \frac{a}{b} = 0$ .

For  $f(x) = \frac{1}{x}$ , as  $x \rightarrow \infty$  the numerator stays constant at 1 and the denominator grows larger in magnitude and positive. Therefore we have a positive constant being divided by an increasingly large positive number. The result of this fraction will be an increasingly small, positive number. Therefore,  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ .

Now, as  $x \rightarrow -\infty$ , again the numerator stays constant at 1, but the denominator grows larger in magnitude and negative. So, we have a positive constant being divided by an increasingly large negative number. The result of this fraction will be an increasingly small, negative number. This means we have  $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$ .

## 2.1 Horizontal Asymptotes

For any given function  $f(x)$  if either of the following limits exists, then  $f(x)$  will have a **horizontal asymptote** at  $y = L$ .

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L$$

*Note: A horizontal asymptote can only occur when the limit at infinity yields a finite number.*

**Example 6:** Let's again revisit the function of  $f(x) = \frac{1}{x}$ .

We just found that  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$  and  $\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$ . From this we are able to conclude that  $f(x) = \frac{1}{x}$  has a horizontal asymptote at  $y = 0$ .

**Example 7:** Find the horizontal asymptotes of the following functions. [*Hint: Consider the limit laws.*]

- $f(x) = \frac{-2}{x}$

- $f(x) = \frac{3}{x} + 5$

## 2.2 Infinite Limits at Infinity

If a function  $f(x)$  increases or decreases without bound as  $x$  increases or decreases without bound, then we have an infinite limit at infinity. These are denoted by:

$$\lim_{x \rightarrow \infty} f(x) = \pm\infty \text{ or } \lim_{x \rightarrow -\infty} f(x) = \pm\infty$$

Here are some examples of infinite limits at infinity:

- $\lim_{x \rightarrow \infty} x = \infty$

- $\lim_{x \rightarrow \infty} e^x = \infty$

- $\lim_{x \rightarrow -\infty} x^2 = \infty$

- $\lim_{x \rightarrow -\infty} x^3 = -\infty$

*Note: In the case of infinite limits at infinity there exists no horizontal asymptotes.*

## 2.3 Limits at Infinity of Powers and Polynomials

Let  $n$  be a positive integer and let  $p$  be the polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$$

where  $a_n \neq 0$ .

- If  $n$  is even:  $\lim_{x \rightarrow \pm\infty} x^n = \infty$
- If  $n$  is odd:  $\lim_{x \rightarrow \infty} x^n = \infty$  and  $\lim_{x \rightarrow -\infty} x^n = -\infty$
- $\lim_{x \rightarrow \pm\infty} \frac{1}{x^n} = 0$
- $\lim_{x \rightarrow \pm\infty} p(x) = \infty$  or  $-\infty$ , depending on the degree (odd or even?) of the polynomial and the sign of the leading coefficient  $a_n$  (positive or negative).

$$- \lim_{x \rightarrow \pm\infty} p(x) = \lim_{x \rightarrow \pm\infty} a_n x^n = a_n \cdot \lim_{x \rightarrow \pm\infty} x^n$$

**Example 8:** Evaluate the following:

- $\lim_{x \rightarrow -\infty} (7x^5 - 4x^3 + 2x - 9) = \lim_{x \rightarrow -\infty} 7x^5 = 7 \cdot \lim_{x \rightarrow -\infty} x^5 = 7 \cdot -\infty = -\infty$
- $\lim_{x \rightarrow \infty} (8x^2 + 3x - 5x^3)$
- $\lim_{x \rightarrow -\infty} (17x^3 - 4x^9 - 5x + 1)$
- $\lim_{x \rightarrow -\infty} \frac{-2}{x^3}$
- $\lim_{x \rightarrow \infty} 18$

## 2.4 Limits at Infinity for Rational Functions

In general, for rational functions, we can evaluate limits at infinity by dividing each term in the expression by **the highest power of  $x$  in the denominator**, simplifying, and then evaluating the limit (for each term).

**Example 9:** Evaluate  $\lim_{x \rightarrow \infty} \frac{3x + 2}{x^2 - 1}$

Since this is a limit at infinity, direct substitution is not a valid approach. However, given this is a rational function, we can divide each term by the highest power in the denominator. For this function, the highest power in the denominator is  $x^2$ . So, we have the following:

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{3x + 2}{x^2 - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{3x}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{3}{x} + \frac{2}{x^2}}{1 - \frac{1}{x^2}} \\ &= \frac{\lim_{x \rightarrow \infty} \frac{3}{x} + \lim_{x \rightarrow \infty} \frac{2}{x^2}}{\lim_{x \rightarrow \infty} 1 - \lim_{x \rightarrow \infty} \frac{1}{x^2}} \\ &= \frac{0 + 0}{1 - 0} \\ &= 0 \end{aligned}$$

For the example, since we found that a limit at infinity led to a finite value, we are able to say  $f(x) = \frac{3x + 2}{x^2 - 1}$  has a horizontal asymptote at  $y = 0$ .

**Example 10:** Evaluate  $\lim_{x \rightarrow \infty} \frac{x + x^3 - 8x^4}{2x^4 + x^2 - 1}$

**Example 11:** Evaluate  $\lim_{x \rightarrow -\infty} \frac{x^5 + 4x^2 - 5}{x^3 + 6x}$