Rates of Change as Limits & The Definition of the Derivative; Differentiability

MAC 2311

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1 Rates of Change as Limits

We begin this section by revisiting rates of change. Recall that the **average rate of change** of a function f on the interval $[a, x]$ is the **slope of the secant line** between the two points x and a.

$$ARoC = m_{secant} = \frac{f(x) - f(a)}{x - a}$$

To find the **instantaneous rate of change** of f at a single point a, we find the **slope of the tangent line** at $x = a$. This can be accomplished by taking the limit of the slope of secant lines between the points x and a as x gets closer to a.

$$IRoC = m_{tangent} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{1}$$

Another way of writing an interval between two points is $[a, a + h]$, where h represents the change in the x-values. So, the expressions for average and instantaneous rates of change becomes:

$$ARoC = m_{secant} = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

$$IRoC = m_{tangent} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

We now have two ways of finding the instantaneous rate of change at a single point $x = a$ by using limits [Equations (1) and (2) above]. This also means we have two ways of calculating the slope of the tangent line at a point $x = a$. So, we are able to also find the **equation for the tangent line** to the graph of $f(x)$ at $x = a$. We can do so in point-slope form by:

$$y - f(a) = m_{tangent}(x - a)$$

Example 1: Let $f(x) = x^2 - 5x$.

- Find the slope of the tangent line to the graph of $f(x)$ at $x = 1$.

  Let's use Equation (2) for finding m_{tan}. Since we are finding the slope of the tangent at $x = 1$, this means in the equation $a = 1$. So, we have:

$$
\begin{aligned}
m_{tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\
&= \lim_{h\to 0} \frac{[(1+h)^2 - 5(1+h)] - (-4)}{h} \\
&= \lim_{h\to 0} \frac{1 + 2h + h^2 - 5 - 5h + 4}{h} \\
&= \lim_{h\to 0} \frac{h^2 - 3h}{h} \\
&= \lim_{h\to 0} \frac{h(h-3)}{h} \\
&= \lim_{h\to 0} (h-3) \\
&= -3
\end{aligned}
$$

- Find the equation of the tangent line at $x = 1$.

$$
\begin{aligned}
y - f(a) &= m_{tan}(x - a) \\
y - f(1) &= m_{tan}(x - 1) \\
y - (-4) &= -3(x - 1) \\
y + 4 &= -3(x - 1) \leftarrow \textit{point-slope form} \\
y &= -3x - 1 \leftarrow \textit{slope-intercept form}
\end{aligned}
$$

Example 2: Let $f(x) = \dfrac{3}{x}$. Using Equation (1), find the slope and equation of the tangent line to the graph of $f(x)$ at $x = 2$.

Example 3: Find the equation of the tangent line to the graph of $f(x) = x^3 + 4$ at $x = -1$.

2 The Derivative Function

In the last section we showed that the limit: $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ can be interpreted as the slope of the tangent line to the curve $y = f(x)$ at the point $x = a$. Let's say we wanted to know the slope of the tangent line at any given point, x, on the curve. This is where what's known as the **derivative function** comes into play.

2.1 Limit Definition of the Derivative

The **derivative** of the function $f(x)$ with respect to x is the function $f'(x)$ defined by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative of $f(x)$ represents the slope of the curve (slope of tangent line) at any given point, x, on $f(x)$.

2.2 Derivative Notation

For $y = f(x)$, the notation for the derivative function includes:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}[f(x)]$$

For a specified point, $x = a$, the derivative of a function $y = f(x)$ at $x = a$ can be denoted as:

$$f'(a) = y'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}[f(x)]\right|_{x=a}$$

Example 4: Using the limit definition of the derivative, find the derivative of $f(x) = 3x^2 - 7x + 2$.

$$\begin{aligned}
f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{[3(x+h)^2 - 7(x+h)+2] - [3x^2 - 7x + 2]}{h} \\
&= \lim_{h\to 0}\frac{[3(x^2+2xh+h^2) - 7(x+h)+2] - [3x^2 - 7x + 2]}{h} \\
&= \lim_{h\to 0}\frac{3x^2+6xh+3h^2-7x-7h+2-3x^2+7x-2}{h} \\
&= \lim_{h\to 0}\frac{6xh+3h^2-7h}{h} \\
&= \lim_{h\to 0}\frac{h(6x+3h-7)}{h} \\
&= \lim_{h\to 0}(6x+3h-7) \\
&= 6x+3(0)-7 \\
&= 6x-7
\end{aligned}$$

Example 5: Using the limit definition of the derivative, find the derivative of $f(x) = \dfrac{5}{x+1}$.

Example 6: Using the limit definition of the derivative, find the derivative of $f(x) = 8 - 2x$.

2.3 The Derivative at a Point

Using the limit definition of the derivative, it follows that the **derivative of a function evaluated at a point** $x = a$, $f'(a)$, is found by:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Note that this is the same equation we used to find the **slope of a tangent line at a point** $x = a$ [See: Equation (2)]. Therefore,

$$m_{tangent} = f'(a)$$

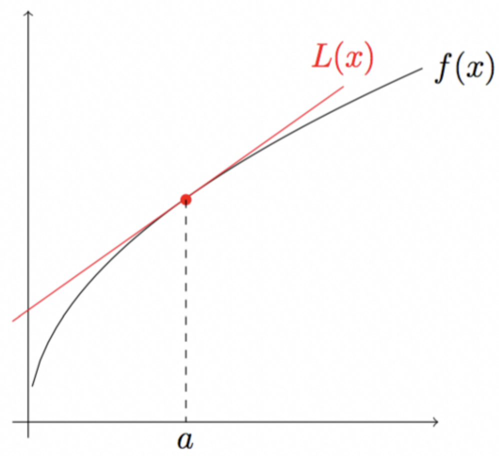
This also means that we can write the **equation of a tangent line** to the curve $y = f(x)$ at the point $x = a$ as:

$$y - f(a) = f'(a)(x - a)$$

Example 7: Find an equation of the tangent line to the curve $f(x) = 2\sqrt{x}$ at $x = 9$.

2.4 Linear Approximation

In "real life", we often have to approximate complicated functions using simpler ones that give us "good enough" accuracy. In this section, we are going to return to our basic understanding of the derivative in order to learn how to make approximations for functions. Recall that if we had a function $f(x)$ at a point $x = a$, then the derivative of f at a represented the slope of the line tangent to f at a:



The tangent line has been labeled $L(x)$ in the figure above. We know how to calculate the equation of the line using the point-slope equation:

$$L(x) - f(a) = f'(a)(x - a)$$

Referring the figure above, we can make an important observation: when we are near a, both $f(x)$ and $L(x)$ are nearly the same. In other words, $f(x)$ is **approximated** by $L(x)$. Therefore, we call the function $L(x)$ the **linear approximation** (or **linearization**) of $f(x)$.

Linear Approximation to f at a Suppose f is differentiable on an interval I containing the point a. The linear approximation to f at a is the linear function:

$$L(x) = f(a) + f'(a)(x - a).$$

Note: The further x is away from a, the more the accuracy decreases.

Example For $f(x) = \frac{x}{x+1}$, write the linear approximation at $a = 1$ and estimate $f(1.1)$.

Example Find the linearization of the function $f(x) = \sqrt{x}$ at $x = 16$ and use it to approximate $\sqrt{15.9}$. Note that the value of $\sqrt{15.9} \approx 3.9874804075$ per the calculator.

Example Use a linear approximation to estimate $\sqrt[3]{8.2}$. Choose a value of a to produce a small error.

3 Differentiability

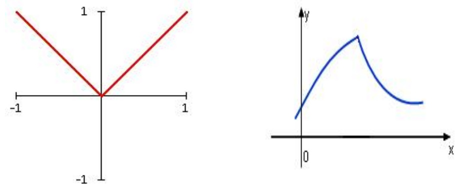
If the derivative, $f'(x)$, exists for any point or interval on $f(x)$, then we say that $f(x)$ is **differentiable** at that point or on that interval.

Theorem: If $f(x)$ is differentiable at $x = a$, then $f(x)$ is guaranteed to be continuous at $x = a$.

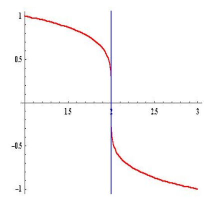
Note: The converse of this theorem is not true. In other words, it is possible for a function to be continuous at a point, but not differentiable at that point.

There are four cases where a function will be **non-differentiable**, or where the derivative will not exist.

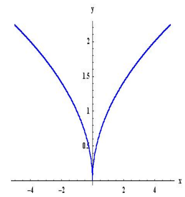
1. **Corner/Sharp Turn:** A corner or sharp turn can be seen on the graph of a function. At this point where the corner or sharp turn exists, the function will not be differentiable because the one-sided derivatives will differ at this point.



2. **Vertical Tangent:** At a vertical tangent the slope of the tangent line approaches ∞ or $-\infty$ from both sides.



3. **Cusp:** At a cusp the slope of the tangent line approaches ∞ from one side and $-\infty$ from the other.



4. **Discontinuity:** If a function is discontinuous at a point, then the derivative will not exist at that point.