# The Chain Rule; Derivatives of Exponential, Logarithmic & Inverse Trigonometric Functions

MAC 2311

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# 1 The Chain Rule

*So far, we have ways for finding derivatives of combinations of functions including the sum, difference, product, and quotient. In this section, we will discuss how to evaluate derivatives of composite functions.*

## 1.1 The Chain Rule - Leibniz Notation

Given a composite function $y = f(g(x))$, if we let $u = g(x)$, then $y = f(u)$. The derivative of $y$ is then found by:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[y] \cdot \frac{d}{dx}[u] \\ &= \frac{d}{du}[f(u)] \cdot \frac{d}{dx}[g(x)] \end{aligned}$$

Using **Leibniz Notation for the chain rule** can be thought of in the following steps:

Given a composite function $y = f(g(x))$, in order to find $\frac{dy}{dx}$,

1. Identify the outer function, $f$, and the inner function, $g$.
2. Let $u = g(x)$ and $y = f(u)$.
3. Find the product $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
4. Put $u$ back in terms of $x$.

**Example 1:** Find the derivative of $y = \cos(x^3)$.

We start by noticing that $y$ is a composite function made up of the two functions, $f(x) = \cos(x)$ and $g(x) = x^3$. So,

$$y = f(g(x)) = \cos(x^3)$$

We'll let $u = g(x)$ and express $y$ as $f(u)$:

$$\begin{aligned} u &= g(x) = x^3 \\ y &= f(u) = \cos(u) \end{aligned}$$

Let's proceed by finding the product for $\dfrac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[\cos(u)] \cdot \frac{d}{dx}[x^3] \\ &= -\sin(u) \cdot 3x^2 \\ &= -\sin(x^3) \cdot 3x^2 \\ &= -3x^2 \sin(x^3) \end{aligned}$$

**Example 2:** Using Leibniz Notation, find the derivative of the following:

- $y = (4 + 5x)^6$

- $y = \sqrt{\tan x}$

## 1.2 The Chain Rule - Prime Notation

Given a composite function $y = (f \circ g)(x) = f(g(x))$, the derivative of this function is:

$$y' = f'(g(x)) \cdot g'(x)$$

Using **Prime Notation for the chain rule** can be thought of in the following steps:

Given a composite function $y = (f \circ g)(x) = f(g(x))$, in order to find $y'$,

1. Identify the outer function, $f(x)$, and the inner function, $g(x)$.
2. Take the derivative of the outer function, $f'(x)$, and evaluate it at the inner function: $f'(g(x))$.
3. Multiply by the derivative of the inner function: $f'(g(x)) \cdot g'(x)$

**Example 3:** Find the derivative of $y = \sin^2(x)$.

It may help to rewrite $y$ so that it is easier to see how the function is composed.

$$y = \sin^2(x) = (\sin x)^2$$

Here, we notice that $y = f(g(x)) = (\sin x)^2$ is a composite function made up of the two functions $f(x) = x^2$ and $g(x) = \sin x$.

Now, let's take take the derivative of the outer function, $f(x) = x^2$:

$$f'(x) = 2x$$

and evaluate it at the inner function, $g(x) = \sin x$:

$$f'(g(x)) = f'(\sin x) = 2 \sin x$$

Finally, let's multiply the result by the derivative of the inner function:

$$\begin{aligned} g'(x) &= \cos x \\ f'(g(x)) \cdot g'(x) &= 2 \sin x \cdot \cos x \end{aligned}$$

So, our derivative of $y = \sin^2(x)$ is:

$$y' = f'(g(x)) \cdot g'(x) = 2 \sin x \cos x$$

**Example 4:** Using Prime Notation, find the derivative of $y = \sec(x^2 + 3x)$.

**Example 5:** Find the derivative of the following:

- $h(x) = \dfrac{1}{(2x^3 + x^2 + 8)^5}$

- $y = x^4 \cos(4x^2)$

## 1.3 Composition of 3 or More Functions

When a function is composed of three or more functions, we must do the chain rule repeatedly to find the derivative. For example, if we are given that

$$y = f(g(h(x)))$$

Then,

$$\frac{dy}{dx} = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

**Example 6:** Find the derivative of $y = \sqrt[4]{\sin(3x - 2)}$.

# 2 Derivatives of Exponential, Logarithmic, & Inverse Trigonometric Functions

## 2.1 Derivatives of Exponential Functions

- $\frac{d}{dx}[b^x] = \ln(b)b^x$
- $\frac{d}{dx}[e^x] = e^x$

**Example 7:** Find the derivative of $f(x) = 3^x - 5e^x$.

**Example 8:** Find the derivative of $y = e^{\cot x}$.

## 2.2 Derivatives of Logarithmic Functions

- $\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$
- $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$

**Example 9:** Find the derivative of $g(x) = 3x^2 \log_3(x)$.

**Example 10:** Find the derivative of $y = \ln(x^2 + 5)$.

## 2.3 Derivatives of Inverse Trigonometric Functions

- $\dfrac{d}{dx}\left[\sin^{-1} x\right] = \dfrac{1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\dfrac{d}{dx}\left[\cos^{-1} x\right] = \dfrac{-1}{\sqrt{1-x^2}}$, for $-1 < x < 1$
- $\dfrac{d}{dx}\left[\tan^{-1} x\right] = \dfrac{1}{1+x^2}$, for $-\infty < x < \infty$
- $\dfrac{d}{dx}\left[\cot^{-1} x\right] = \dfrac{-1}{1+x^2}$, for $-\infty < x < \infty$
- $\dfrac{d}{dx}\left[\sec^{-1} x\right] = \dfrac{1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$
- $\dfrac{d}{dx}\left[\csc^{-1} x\right] = \dfrac{-1}{|x|\sqrt{x^2-1}}$, for $|x| > 1$

**Example 11:** Find the derivative of $y = \dfrac{1+x^2}{\cot^{-1} x}$.

**Example 12:** Find the derivative of $h(x) = \sec^{-1}(2x)$.