GESTURE, STRUGGLE, AND PROGRESS: EXAMPLES FROM THE UNDERGRADUATE TOPOLOGY CLASSROOM

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Picture a circle. Any two points on the circle connected by any diameter are called “antipodal points.”

What shape is obtained by gluing every point on the circle to its antipodal point?
What do you think?
Think of the circle as a rubber band.

1. Twist the rubber band in the middle and glue the cross point ("identify" a=a').

2. Fold the resulting circles onto one another and glue!
Why study gestures?
Identifying gestures

Types of gesture:
• Beat
• Deictic
• Iconic (IC)
• Metaphoric (MP)
Results from neurology

• Rizzolatti & Craighero (2004) –
  ➢ mirror neurons fire when an action is produced and when an action is observed
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• Straube, Green, Bromberger, & Kircher (2011) –
  ➢ Gesture processing: perception of iconic gestures activates a subset of the regions activated by metaphoric gestures
Results from neurology

  - mirror neurons fire when an action is produced and when an action is observed
- Straube, Green, Bromberger, & Kircher (2011) –
  - Gesture processing: perception of iconic gestures activates a subset of the regions activated by metaphoric gestures
  - Gesture production: beat gestures associated with brain volume in discrete motor timing regions, metaphoric gestures associated with brain volume in metaphoric processing regions
Hostetter & Alibali (2008) –

Gesture-as-Simulated-Action Framework: “gestures emerge from the perceptual and motor simulations that underlie embodied language and mental imagery.” (p. 502)

Recall: gesture processing and production activate the same brain regions.
Results from cognitive science

  - Gestures not only reflect thinking, but feed back and alter thinking
  - Actions and gestures influence thinking
  - Gestures contain perceptual-motor information

\[ \text{Gesture} \quad + \quad \text{Mental representation} \quad \rightarrow \quad \text{New mental representation} \quad \rightarrow \quad \text{Behavior} \]
• Alibali et al. (2011) –
  ➢ “Spontaneous gestures influence strategy choice in problem solving”
  ➢ Participants solved six problems involving gears, e.g., “Imagine [4, 7, 9, 5, 8, 6] gears are arranged in a horizontal line. If you turn the gear on the left clockwise, what would the gear on the right do?”

Results from cognitive science
What does this have to do with math?

- Gestures can give insight into students’ thinking and attention
- Gesture processing and production can alter students’ thinking
Two handy conceptual frameworks

Embodied Cognition

Commognition
What is embodied cognition?

“Many features of cognition are embodied in that they are deeply dependent upon characteristics of the physical body of an agent, such that the agent's beyond-the-brain body plays a significant causal role, or a physically constitutive role, in that agent's cognitive processing.”

(Stanford Encyclopedia of Philosophy, 2017)
What is embodied cognition?

Three roles for the body:

- **Body as Constraint:** an agent's body functions to significantly constrain the nature and content of the representations processed by that agent's cognitive system.
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Three roles for the body:

- **Body as Constraint**: an agent's body functions to significantly constrain the nature and content of the representations processed by that agent's cognitive system.

- **Body as Distributor**: an agent's body functions to distribute computational and representational load between neural and non-neural structures.
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Three roles for the body:

- **Body as Constraint**: an agent's body functions to significantly constrain the nature and content of the representations processed by that agent's cognitive system.

- **Body as Distributor**: an agent's body functions to distribute computational and representational load between neural and non-neural structures.

- **Body as Regulator**: an agent's body functions to regulate cognitive activity over space and time, ensuring that cognition and action are tightly coordinated.
What is commognition?

- Thinking is the individualized form of the activity of communicating.
- Thinking is *not* a self-sustained process separate from communication but is an act of communication with oneself.
- This self-communication does not have to be audible or visible and does not have to be in words.
- Cognitive processes and interpersonal communication processes are different manifestations of basically the same phenomenon. (Sfard, 2008, pp. 82-83)
Gestures in proof writing

• How do undergraduates use gestures and diagrams to support their proof construction in topology?

• Semiotic bundle
  “The novelty of the semiotic bundle...is that it allows us to describe the multimodal semiotic activity of subjects in a holistic way as a dynamic production and transformation of various signs and of their relationships. In particular, it properly frames the role of gestures in mathematical activities.” (Arzarello, Paoloa, Robutti, and Sabena, 2009, p. 100)
Gestures in proof writing

- **Key ideas** in proof
  - **Heuristic idea** – based on informal understandings and provides a sense of understanding (convincing yourself)
  - **Procedural idea** – based on logic and formal manipulations to provide a sense of conviction (convincing others)

- “A **key idea** is an heuristic idea which one can map to a formal proof with appropriate sense of rigor.” (Raman, 2003, p. 323) (convincing yourself and others)
Gestures in proof writing

• The task

- **Definition:** Given a topological space \((X, \mathcal{T})\), a subset \(A\) of \(X\) is *dense* in \(X\) if the (topological) closure of \(A\) is equal to \(X\).

- **Prove:** Let \((X, \mathcal{T})\) be a topological space, and let \(A\) be a subset of \(X\). If for each open set \(O \in \mathcal{T}\) we have \(A \cap O \neq \emptyset\), then \(A\) is dense in \(X\).
Gestures in proof writing
Gestures in proof writing

I can’t really show it with a picture because I can’t draw a dashed line over a … solid line, but we have $X$ on the outside and then we have the set $A$ which is represented by the dashed, which I wish I could get closer to this, but I can’t. So, if we had the closure of $A$, then it would just be the same as that solid line. So then if you take any open set anywhere, there has to be some kind of intersection with $A$. So if it wasn’t … if the intersection could be … the empty set – You’ve got $X$ here, and $A$ here, and you could have an open set here, and their intersection would be the empty set. But then this closure wouldn’t be equal to $X$. I get it conceptually I think, but I’m not sure how to prove it.
Gestures in proof writing
Gestures and struggle

- **Productive Struggle**
  - “[E]ffort to make sense of mathematics, to figure something out that is not immediately apparent” (Hiebert & Grouws, 2007, p. 387)

- **Unproductive Struggle**
  - “[N]eedless frustration or extreme levels of challenge created by nonsensical or overly difficult problems... [or] the feelings of despair that some students can experience when little of the material makes sense” (ibid.)
Gestures and struggle

- **Zone of proximal development**

  - “[T]he distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance, or in collaboration with more capable peers” (Vygotsky, 1978, p. 86)
Gestures and struggle

• The task
  ➢ **Prove:** Let $f : S \rightarrow T$ be a function, and let $\{U_i\}_{i \in I}$ be a family of subsets of $T$. Prove that
    • $f^{-1}(\bigcap_{i \in I} U_i) = \bigcap_{i \in I} f^{-1}(U_i)$.

• Started proof by element chasing
  ➢ Immediate confusion over notation, where $x$ should lie
  ➢ I recommended drawing a diagram
Gestures and struggle
Gestures and struggle

Let $f: S \to T$, and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a family of subsets of $T$. Prove that $f^{-1}(\bigcup_{i \in I} U_i) = \bigcap_{i \in I} f^{-1}(U_i)$. Let $x \in f^{-1}(\bigcap_{i \in I} U_i)$. Then $x \in f^{-1}(U_i)$ for all $i \in I$. Thus, $f(x) \in U_i$ for all $i \in I$. Hence, $x \in \bigcup_{i \in I} U_i$. Therefore, $f^{-1}(\bigcup_{i \in I} U_i) \subseteq \bigcap_{i \in I} f^{-1}(U_i)$. Conversely, let $x \in \bigcap_{i \in I} f^{-1}(U_i)$. Then $x \in f^{-1}(U_i)$ for all $i \in I$. Thus, $f(x) \in U_i$ for all $i \in I$. Hence, $x \in \bigcup_{i \in I} U_i$. Therefore, $\bigcap_{i \in I} f^{-1}(U_i) \subseteq f^{-1}(\bigcup_{i \in I} U_i)$. Hence, $f^{-1}(\bigcup_{i \in I} U_i) = \bigcap_{i \in I} f^{-1}(U_i)$. 
Gestures and struggle

Gestures accompanied progress

No gestures during unproductive struggle
Some additional reading


Thank you!

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