Trace Formulas Applied to the Riemann ζ-Function

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Abstract We use a spectral theory perspective to reconsider properties of the Riemann zeta function. In particular, new integral representations are derived and used to present its value at odd positive integers.

Keywords Dirichlet Laplacian · Trace class operators · Trace formulas · Riemann zeta function · Spectral theory

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Ş. Kuru et al. (eds.), *Integrability, Supersymmetry and Coherent States*, CRM Series in Mathematical Physics, https://doi.org/10.1007/978-3-030-20087-9_8

1 Introduction

Spectral zeta functions associated with eigenvalue problems of (partial) differential operators are of relevance in a wide array of topics [3–8, 10, 11, 16, 18, 25, 26, 28]. As an example consider the Dirichlet boundary value problem

$$-\Delta_D f = -f'', \quad f(0) = f(1) = 0, \tag{1.1}$$

where $-\Delta_D$ denotes the Dirichlet Laplacian in the Hilbert space $L^2((0,1); dx)$ (cf. (2.12)), with purely discrete and simple spectrum,

$$\sigma(-\Delta_D) = \{\lambda_k = (k\pi)^2\}_{k\in\mathbb{N}}. \tag{1.2}$$

In particular, the spectral zeta function associated with $-\Delta_D$,

$$\zeta(z; -\Delta_D) = \sum_{k\in\mathbb{N}} \lambda_k^{-z} = \pi^{-2z}\zeta(2z), \quad \mathrm{Re}(z) > 1/2, \tag{1.3}$$

is basically given by the Riemann zeta function

$$\zeta(z) = \sum_{k\in\mathbb{N}} k^{-z}, \quad \mathrm{Re}(z) > 1. \tag{1.4}$$

This elementary and well-known observation identifies the Riemann zeta function as a spectral zeta function and hence spectral theoretic techniques for their analysis can be applied to it. This is the perspective taken in this article. In Sect. 2 we briefly review representations for spectral zeta functions as derived in [13] and we apply them to the zeta function of Riemann. New integral representations for the Riemann zeta function are found and the well-known properties, namely values at even negative and positive integers are easily reproduced. In addition, we derive new representations for the value of the Riemann zeta function at positive odd integers. Typical examples we derive are

$$\begin{gathered}\zeta(z) = \sin(\pi z/2)\pi^{z-1}\int_0^\infty ds\, s^{-z}[\coth(s) - \coth_n(s)], \\ \mathrm{Re}(z) \in (\max(1, 2n), 2n+2),\ n \in \mathbb{N}_0,\end{gathered} \tag{1.5}$$

where

$$\begin{aligned}\coth_0(z) &= \frac{1}{z}, \quad z \in \mathbb{C}\backslash\{0\}, \\ \coth_n(z) &= \frac{1}{z} + \sum_{k=1}^{n} \frac{2^{2k}B_{2k}}{(2k)!} z^{2k-1}, \quad z \in \mathbb{C}\backslash\{0\},\ n \in \mathbb{N},\end{aligned} \tag{1.6}$$

with B_m the Bernoulli numbers (cf. (A.30)–(A.33)), implying

$$\zeta(3) = -\pi^2 \int_0^\infty ds\, s^{-3}[\coth(s) - (1/s) - (s/3)], \tag{1.7}$$

$$\zeta(5) = \pi^4 \int_0^\infty ds\, s^{-5}\big[\coth(s) - (1/s) - (s/3) + \big(s^3/45\big)\big], \tag{1.8}$$

$$\zeta(7) = -\pi^6 \int_0^\infty ds\, s^{-7}\big[\coth(s) - (1/s) - (s/3) + \big(s^3/45\big) - \big(2s^5/945\big)\big], \quad \text{etc.} \tag{1.9}$$

Finally, the Appendix summarizes known results about the Riemann zeta function putting the results we found in some perspective.

2 Computing Traces and the Riemann ζ-Function

After a brief discussion of spectral zeta functions associated with self-adjoint operators with purely discrete spectra, we turn to applications of spectral trace formulas to the Riemann zeta function.

We start by following the recent paper [13] and briefly discuss spectral ζ-functions of self-adjoint operators S with a trace class resolvent (and hence a purely discrete spectrum).

Below we will employ the following notational conventions: A separable, complex Hilbert space is denoted by $\mathcal{H}$, $I_{\mathcal{H}}$ represents the identity operator in $\mathcal{H}$; the resolvent set and spectrum of a closed operator T in $\mathcal{H}$ are abbreviated by $\rho(T)$ and $\sigma(T)$, respectively; the Banach space of trace class operators on $\mathcal{H}$ is denoted by $\mathcal{B}_1(\mathcal{H})$, and the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ is abbreviated by $\operatorname{tr}_{\mathcal{H}}(A)$.

Hypothesis 2.1 *Suppose S is a self-adjoint operator in $\mathcal{H}$, bounded from below, satisfying*

$$(S - zI_{\mathcal{H}})^{-1} \in \mathcal{B}_1(\mathcal{H}) \tag{2.2}$$

for some (and hence for all) $z \in \rho(S)$. We denote the spectrum of S by $\sigma(S) = \{\lambda_j\}_{j\in J}$ (with $J \subset \mathbb{Z}$ an appropriate index set), with every eigenvalue repeated according to its multiplicity.

Given Hypothesis 2.1, the spectral zeta function of S is then defined by

$$\zeta(z; S) = \sum_{\substack{j\in J \\ \lambda_j \neq 0}} \lambda_j^{-z} \tag{2.3}$$

for $\operatorname{Re}(z) > 0$ sufficiently large such that (2.3) converges absolutely.

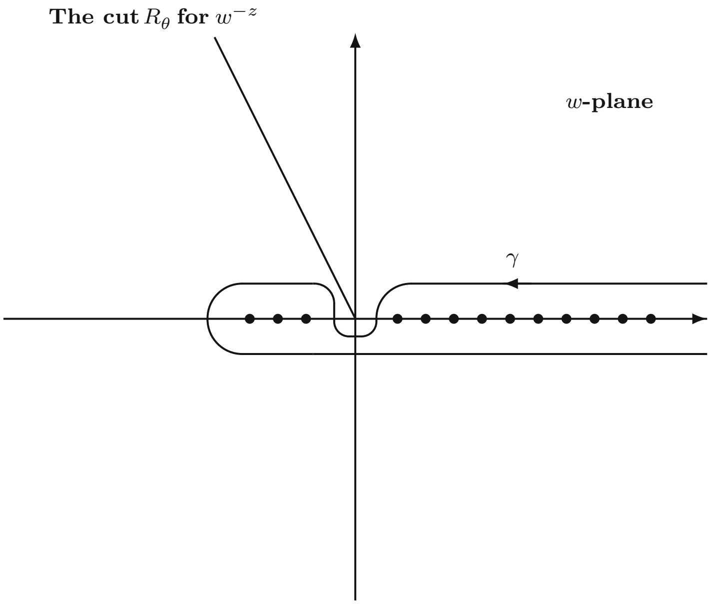

Fig. 1 Contour γ in the complex w-plane

Next, let $P(0; S)$ be the spectral projection of S corresponding to the eigenvalue 0 and denote by $m(\lambda_0; S)$ the multiplicity of the eigenvalue λ_0 of S, in particular,

$$m(0; S) = \dim(\ker(S)). \tag{2.4}$$

In addition, we introduce the simple contour γ encircling $\sigma(S)\backslash\{0\}$ in a counterclockwise manner so as to dip under (and hence avoid) the point 0 (cf. Fig. 1). In fact, following [20] (see also [19]), we will henceforth choose as the branch cut of w^{-z} the ray

$$R_\theta = \big\{w = te^{i\theta}\big|t \in [0, \infty)\big\} \quad \theta \in (\pi/2, \pi), \tag{2.5}$$

and note that the contour γ avoids any contact with R_θ (cf. Fig. 1).

Lemma 2.6 *In addition to Hypothesis 2.1 and the counterclockwise oriented contour γ just described (cf. Fig. 1), suppose that $\big|\mathrm{tr}_{\mathcal{H}}\big((S-zI_{\mathcal{H}})^{-1}[I_{\mathcal{H}}-P(0; S)]\big)\big|$ is polynomially bounded with respect to z on γ. Then*

$$\zeta(z; S) = -(2\pi i)^{-1} \oint_\gamma dw\, w^{-z}\big[\,\mathrm{tr}_{\mathcal{H}}\big((S - wI_{\mathcal{H}})^{-1}\big) + w^{-1}m(0; S)\big] \tag{2.7}$$

for $\mathrm{Re}(z) > 0$ *sufficiently large.*

We note in passing that one could also use a semigroup approach via

$$\begin{aligned}\zeta(z;S) &= \Gamma(z)^{-1}\int_0^\infty dt\, t^{z-1}\,\mathrm{tr}_{\mathcal{H}}\big(e^{-tS}[I_{\mathcal{H}} - P(0;S)]\big)\\ &= \Gamma(z)^{-1}\int_0^\infty dt\, t^{z-1}\big[\mathrm{tr}_{\mathcal{H}}\big(e^{-tS}\big) - m(0;S)\big],\end{aligned} \tag{2.8}$$

for $\mathrm{Re}(z) > 0$ sufficiently large.

It is natural to continue the computation leading to (2.7) and now deform the contour γ so as to "hug" the branch cut R_θ, but this requires the right asymptotic behavior of $\mathrm{tr}_{\mathcal{H}}\big((S - wI_{\mathcal{H}})^{-1}[I_{\mathcal{H}} - P(0;S)]\big)$ as $|w| \to \infty$ as well as $|w| \to 0$. This applies, in particular, to cases where S is strictly positive and one thus chooses the branch cut along the negative axis, that is, it employs the cut R_π, where

$$R_\pi = (-\infty, 0], \tag{2.9}$$

and choosing the contour γ to encircle R_π clockwise. This renders (2.7) into the following expression:

$$\begin{aligned}\zeta(z;S) &= \frac{\sin(\pi z)}{\pi}\int_0^\infty dt\, t^{-z}\,\mathrm{tr}_{\mathcal{H}}\big((S + tI_{\mathcal{H}})^{-1}\big)\\ &= \mathrm{tr}_{\mathcal{H}}\bigg(\frac{\sin(\pi z)}{\pi}\int_0^\infty dt\, t^{-z}(S + tI_{\mathcal{H}})^{-1}\bigg) = \mathrm{tr}_{\mathcal{H}}\big(S^{-z}\big),\end{aligned} \tag{2.10}$$

employing the fact,

$$S^{-z} = \frac{\sin(\pi z)}{\pi}\int_0^\infty dt\, t^{-z}(S + tI_{\mathcal{H}})^{-1}, \quad \mathrm{Re}(z) \in (0,1), \tag{2.11}$$

whenever $S \geq 0$ in $\mathcal{H}$, with $\ker(S) = \{0\}$ (see, e.g., [15, Proposition 3.2.1 d)]).

Note While (2.11) is rigorous, the manipulations in (2.10) are formal and subject to appropriate convergence and trace class hypotheses which will affect the possible range of $\mathrm{Re}(z)$. ◇

These hypotheses are easily shown to be satisfied when discussing the one-dimensional Dirichlet Laplacian $-\Delta_D$ in $L^2((0,1);dx)$,

$$\begin{aligned}&-\Delta_D = -\frac{d^2}{dx^2},\\ &\mathrm{dom}(-\Delta_D) = \big\{u \in L^2((0,1);dx)\,\big|\, u, u' \in AC_{loc}([0,1]);\ u(0) = 0 = u(1);\\ &\qquad u'' \in L^2((0,1);dx)\big\}\end{aligned} \tag{2.12}$$

(here $AC([0,1])$ denotes the set of absolutely continuous functions on $[0,1]$).

Recalling Riemann's celebrated zeta function (see the Appendix for more details),

$$\zeta(z) = \sum_{k\in\mathbb{N}} k^{-z}, \quad z \in \mathbb{C},\ \mathrm{Re}(z) > 1. \tag{2.13}$$

we start with the following result.

Lemma 2.14 *Let* $\mathrm{Re}(z) \in (1, 2)$*, then*

$$\zeta(z) = \sin(\pi z/2)\pi^{z-1} \int_0^\infty ds\, s^{-z-1}[s\coth(s) - 1]. \tag{2.15}$$

Proof Since the eigenvalue problem for $-\Delta_D$ reads

$$-\Delta_D u_k = \lambda_k u_k, \quad k \in \mathbb{N}, \tag{2.16}$$

with

$$u_k(x) = 2^{1/2}\sin(k\pi x),\ \|u_k\|_{L^2((0,1);dx)} = 1, \quad \lambda_k = (\pi k)^2, \quad k \in \mathbb{N}, \tag{2.17}$$

and all eigenvalues λ_k, $k \in \mathbb{N}$, are simple, (2.10) works as long as $\mathrm{Re}(z) \in ((1/2), 1)$ and one obtains (see also [7, p. 94])

$$\begin{aligned}
\zeta(z; -\Delta_D) &= \mathrm{tr}_{L^2((0,1);dx)}\big((-\Delta_D)^{-z}\big) = \sum_{k\in\mathbb{N}} (\pi k)^{-2z} = \pi^{-2z}\zeta(2z) \\
&= \frac{\sin(\pi z)}{\pi}\int_0^\infty dt\, t^{-z}\, \mathrm{tr}_{L^2((0,1);dx)}\big((-\Delta_D + tI_{L^2((0,1);dx)})^{-1}\big) \\
&= \frac{\sin(\pi z)}{\pi}\int_0^\infty dt\, t^{-z}\int_0^1 dx\, t^{-1/2}\big[\sinh(t^{1/2})\big]^{-1}\sinh\big(t^{1/2}x\big) \\
&\qquad \times \sinh\big(t^{1/2}(1-x)\big) \\
&= \frac{\sin(\pi z)}{2\pi}\int_0^\infty dt\, t^{-z-1}\big[t^{1/2}\coth\big(t^{1/2}\big) - 1\big], \quad \mathrm{Re}(z) \in ((1/2), 1).
\end{aligned} \tag{2.18}$$

Here we used

$$\begin{aligned}
&(-\Delta_D - zI_{L^2((0,1);dx)})^{-1}(z, x, x') \\
&\quad = \frac{1}{z^{1/2}\sin(z^{1/2})}\begin{cases} \sin(z^{1/2}x)\sin(z^{1/2}(1-x')), & 0 \le x \le x' \le 1, \\ \sin(z^{1/2}x')\sin(z^{1/2}(1-x)), & 0 \le x' \le x \le 1, \end{cases} \\
&\qquad\qquad z \in \mathbb{C}\backslash\{\pi^2k^2\}_{k\in\mathbb{N}},
\end{aligned} \tag{2.19}$$

and [14, 2.4254]

$$\int^x dt\ \sinh(at+b)\sinh(at+c) \\ = -(x/2)\cosh(b-c) + (4a)^{-1}\sinh(2ax+b+c) + C, \quad a \neq 0. \tag{2.20}$$

Since

$$\left[t^{1/2}\coth\left(t^{1/2}\right) - 1\right] = \begin{cases} O(t), & t \downarrow 0, \\ O\left(t^{1/2}\right), & t \uparrow \infty, \end{cases} \tag{2.21}$$

(2.18) is well-defined for $\mathrm{Re}(z) \in ((1/2), 1)$. Thus, the elementary change of variables $t = s^2$ yields (2.15).

Remark 2.22 Representation (2.15) is suitable to observe the well-known properties

$$\zeta(0) = -1/2, \quad \zeta(-2n) = 0, \ n \in \mathbb{N}. \tag{2.23}$$

To this end, one notes that the restriction $\mathrm{Re}(z) > 1$ results from the $s \to \infty$ behavior of the integrand in (2.15). Explicitly, one has

$$s\coth(s) - 1 \underset{s\to\infty}{=} s - 1 + O\left(e^{-2s}\right), \tag{2.24}$$

from which one infers that

$$\zeta(z) = \sin(\pi z/2)\pi^{z-1}\int_1^\infty ds\, s^{-z-1}(s-1) + E(z) \\ = \sin(\pi z/2)\pi^{z-1}\left[\frac{1}{z-1} - \frac{1}{z}\right] + E(z), \tag{2.25}$$

where $E(\cdot)$ is entire and

$$E(-2n) = 0, \quad n \in \mathbb{N}_0. \tag{2.26}$$

This immediately implies (2.23).

The values at positive even integers, $\zeta(2m)$ for $m \in \mathbb{N}$, are best obtained using representation (2.7); see Remark 2.41. ◇

One can generalize (2.11) as follows:

$$S^{-z} = \frac{\Gamma(m)}{\Gamma(n-z)\Gamma(m-n+z)} S^{m-n}\int_0^\infty dt\, t^{n-1-z}(S + tI_{\mathcal{H}})^{-m}, \\ \mathrm{Re}(z) \in (n-m, n), \ m, n \in \mathbb{N}. \tag{2.27}$$

Formally, this now yields

$$\begin{aligned}\zeta(z;S) &= \frac{\Gamma(m)}{\Gamma(n-z)\Gamma(m-n+z)}\int_0^\infty dt\, t^{n-1-z}\,\mathrm{tr}_{\mathcal{H}}\big(S^{m-n}(S+tI_{\mathcal{H}})^{-m}\big)\\ &= \mathrm{tr}_{\mathcal{H}}\bigg(\frac{\Gamma(m)}{\Gamma(n-z)\Gamma(m-n+z)}S^{m-n}\int_0^\infty dt\, t^{n-1-z}(S+tI_{\mathcal{H}})^{-m}\bigg)\\ &= \mathrm{tr}_{\mathcal{H}}\big(S^{-z}\big).\end{aligned} \tag{2.28}$$

The case $m = n$ appears to be the simplest and yields

$$\zeta(z;S) = \frac{\Gamma(n)}{\Gamma(n-z)\Gamma(z)}\int_0^\infty dt\, t^{n-1-z}\,\mathrm{tr}_{\mathcal{H}}\big((S+tI_{\mathcal{H}})^{-n}\big) \tag{2.29}$$
$$= \mathrm{tr}_{\mathcal{H}}\bigg(\frac{\Gamma(n)}{\Gamma(n-z)\Gamma(z)}\int_0^\infty dt\, t^{n-1-z}(S+tI_{\mathcal{H}})^{-n}\bigg) = \mathrm{tr}_{\mathcal{H}}\big(S^{-z}\big).$$

Note Again, (2.27) is rigorous, but (2.28), (2.29) are subject to "appropriate" convergence and trace class hypotheses. ◇

For $-\Delta_D$, (2.29) indeed works for $n \in \mathbb{N}$ as long as $\mathrm{Re}(z) \in ((1/2), n)$ and one obtains

$$\begin{aligned}\zeta(z;-\Delta_D) &= \mathrm{tr}_{L^2((0,1);dx)}\big((-\Delta_D)^{-z}\big) = \sum_{k\in\mathbb{N}}(\pi k)^{-2z} = \pi^{-2z}\zeta(2z)\\ &= \frac{\Gamma(n)}{\Gamma(n-z)\Gamma(z)}\int_0^\infty dt\, t^{n-1-z}\,\mathrm{tr}_{L^2((0,1);dx)}\big((-\Delta_D+tI_{L^2((0,1);dx)})^{-n}\big).\end{aligned} \tag{2.30}$$

For $n = 2$ this includes $z = 3/2$ and hence leads to a formula for $\zeta(3)$. However, we prefer an alternative approach based on [13, Theorem 3.4 (i)] that applies to $-\Delta_D$ and yields the following results.

Lemma 2.31 *Let* $\mathrm{Re}(z) \in (1, 4)$*, then*

$$\zeta(z) = \frac{\sin(\pi z/2)}{(2-z)}\pi^{z-1}\int_0^\infty ds\, s^{-z-1}\Big[s\coth(s) + s^2[\sinh(s)]^{-2} - 2\Big]. \tag{2.32}$$

In particular,

$$\zeta(2) = \frac{\pi^2}{2}\int_0^\infty ds\, s^{-3}\Big[s\coth(s) + s^2[\sinh(s)]^{-2} - 2\Big] = ... = \pi^2/6, \tag{2.33}$$

$$\zeta(3) = \pi^2\int_0^\infty ds\, s^{-4}\Big[s\coth(s) + s^2[\sinh(s)]^{-2} - 2\Big]. \tag{2.34}$$

Proof Employing [13, Theorem 3.4 (i)],

$$\operatorname{tr}_{L^2((0,1);dx)}\big((-\Delta_D - zI_{L^2((0,1);dx)})^{-1}\big) = -(d/dz)\ln\big(z^{-1/2}\sin\big(z^{1/2}\big)\big),$$
$$z \in \mathbb{C}\backslash\big\{\pi^2k^2\big\}_{k\in\mathbb{N}} \tag{2.35}$$

(see also (A.35)), one confirms that

$$\zeta(2)/\pi^2 = \lim_{z\to 0}\sum_{k\in\mathbb{N}}\big(\pi^2k^2 - z\big)^{-1} = 1/6. \tag{2.36}$$

Actually, setting $z = -t$ in (2.35) yields

$$\begin{aligned}&\operatorname{tr}_{L^2((0,1);dx)}\big((-\Delta_D + tI_{L^2((0,1);dx)})^{-1}\big) = (d/dt)\ln\big(t^{-1/2}\sinh\big(t^{1/2}\big)\big)\\&\quad = (2t)^{-1}\big[t^{1/2}\coth\big(t^{1/2}\big) - 1\big], \quad t > 0,\end{aligned} \tag{2.37}$$

and hence confirms (2.18). Continuing that process, one notes that

$$\begin{aligned}&\operatorname{tr}_{L^2((0,1);dx)}\big((-\Delta_D + tI_{L^2((0,1);dx)})^{-2}\big) = -\big(d^2/dt^2\big)\ln\big(t^{-1/2}\sinh\big(t^{1/2}\big)\big)\\&\quad = \big(4t^2\big)^{-1}\Big[t^{1/2}\coth\big(t^{1/2}\big) + t\big[\sinh\big(t^{1/2}\big)\big]^{-2} - 2\Big], \quad t > 0.\end{aligned} \tag{2.38}$$

Insertion of (2.38) into (2.30) taking $n = 2$ then yields

$$\begin{aligned}\zeta(z; -\Delta_D) &= \operatorname{tr}_{L^2((0,1);dx)}\big((-\Delta_D)^{-z}\big) = \sum_{k\in\mathbb{N}}(\pi k)^{-2z} = \pi^{-2z}\zeta(2z)\\&= \frac{1}{\Gamma(2-z)\Gamma(z)}\int_0^\infty dt\, t^{1-z}\operatorname{tr}_{L^2((0,1);dx)}\big((-\Delta_D + tI_{L^2((0,1);dx)})^{-2}\big)\\&= \frac{\sin(\pi z)}{4\pi(1-z)}\int_0^\infty dt\, t^{-1-z}\Big[t^{1/2}\coth\big(t^{1/2}\big) + t\big[\sinh\big(t^{1/2}\big)\big]^{-2} - 2\Big],\end{aligned} \tag{2.39}$$
$$\operatorname{Re}(z) \in ((1/2), 2).$$

Since

$$\Big[t^{1/2}\coth\big(t^{1/2}\big) + t\big[\sinh\big(t^{1/2}\big)\big]^{-2} - 2\Big] = \begin{cases} O\big(t^2\big), & t \downarrow 0,\\ O\big(t^{1/2}\big), & t \uparrow \infty,\end{cases} \tag{2.40}$$

(2.39) is well-defined for $\operatorname{Re}(z) \in ((1/2), 2)$. Thus, the elementary change of variables $t = s^2$ yields (2.32)–(2.34).

Remark 2.41 Employing (2.35) in (2.7), one finds the representation

$$\zeta(z; -\Delta_D) = (2\pi i)^{-1} \oint_{\gamma} dw\, w^{-z} \frac{1 - w^{1/2}\cot\big(w^{1/2}\big)}{2w}, \tag{2.42}$$

where the counterclockwise contour γ can be chosen to consist of a circle γ_ϵ of radius $\epsilon < \pi$ and straight lines γ_1, respectively γ_2, just above, respectively just below, the negative x-axis. For $z = m$, $m \in \mathbb{N}$, contributions from γ_1 and γ_2 cancel each other and thus

$$\zeta(m; -\Delta_D) = (2\pi i)^{-1} \oint_{\gamma_\epsilon} dw\, w^{-m} \frac{1 - w^{1/2}\cot\big(w^{1/2}\big)}{2w}. \tag{2.43}$$

This integral is easily computed using the residue theorem. From the Taylor series [14]

$$\frac{1 - w^{1/2}\cot\big(w^{1/2}\big)}{2w} = \sum_{k=1}^{\infty} \frac{2^{2k-1}|B_{2k}|}{(2k)!} w^{k-1}, \quad w \in \mathbb{C},\ 0 < |w| < \pi^2 \tag{2.44}$$

(with B_m the Bernoulli numbers, cf. (A.30)–(A.33)), the relevant term is $k = m$ and

$$\zeta(m; -\Delta_D) = \frac{2^{2m-1}|B_{2m}|}{(2m)!}, \tag{2.45}$$

implying Euler's celebrated result,

$$\zeta(2m) = \frac{2^{2m-1}\pi^{2m}|B_{2m}|}{(2m)!}, \quad m \in \mathbb{N}. \tag{2.46}$$

This procedure works in a much more general context and allows for the computation of traces of powers of Sturm–Liouville operators in a straightforward fashion; this will be revisited elsewhere. ◇

Remark 2.47 An elementary integration by parts of the term $s^{-2}\coth(s)$ in (2.33) indeed verifies once more that $\zeta(2) = \pi^2/6$. The same integration by parts in (2.34) fails to render the integral trivial (as it obviously should not be trivial). Indeed,

$$\begin{aligned} &\int_\varepsilon^R ds\, \big\{s^{-2}\coth(s) + s^{-1}[\sinh(s)]^{-2} - 2s^{-3}\big\} \\ &= \int_\varepsilon^R ds\, \big\{ -\big[(d/ds)s^{-1}\big]\coth(s) + s^{-1}[\sinh(s)]^{-2} - 2s^{-3}\big\} \\ &= -s^{-1}\coth(s)\Big|_\varepsilon^R + \int_\varepsilon^R ds\,(-2)s^{-3} \underset{\varepsilon\downarrow 0,\, R\uparrow\infty}{\longrightarrow} = \frac{1}{3}. \end{aligned} \tag{2.48}$$

Applying the same strategy to (2.34) yields

$$\int_{\varepsilon}^{R} ds\,\Big[s^{-3}\coth(s) + s^{-2}[\sinh(s)]^{-2} - 2s^{-4}\Big]$$
$$= \int_{\varepsilon}^{R} ds\,\Big[-(1/2)\big[(d/ds)s^{-2}\big]\coth(s) + s^{-2}[\sinh(s)]^{-2} - 2s^{-4}\Big]$$
$$= -(1/2)s^{-2}\coth(s)\Big|_{\varepsilon}^{R} + \int_{\varepsilon}^{R} ds\,\Big[(1/2)s^{-2}[\sinh(s)]^{-2} - 2s^{-4}\Big], \tag{2.49}$$

and hence the expected nontrivial integral. We note once more that

$$\Big[s\coth(s) + s^{2}[\sinh(s)]^{-2} - 2\Big] \underset{s\downarrow 0}{=} O\big(s^{4}\big), \tag{2.50}$$

rendering (2.34) well-defined. ◇

The following alternative (though, equivalent) approach to $\zeta(z)$ is perhaps a bit more streamlined.

Theorem 2.51 *Let $n \in \mathbb{N}_0$, $0 < \mathrm{Re}(z) < 1$, and*[1] $\mathrm{Re}(2n + 2z) > 1$. *Then*[2]

$$\zeta(2n+2z) = \frac{(-1)^n \pi^{2(n+z)}}{\Gamma(1-z)\Gamma(n+z)} \int_0^{\infty} dt\; t^{-z} \frac{d^n}{dt^n}\Big[(2t)^{-1}\big[t^{1/2}\coth\big(t^{1/2}\big) - 1\big]\Big]$$
$$= \frac{(-1)^n 2^{-n} \pi^{2(n+z)}}{\Gamma(1-z)\Gamma(n+z)} \int_0^{\infty} ds\; s^{1-2z} \left(\frac{1}{s}\frac{d}{ds}\right)^{n} \Big[s^{-2}[s\coth(s) - 1]\Big]. \tag{2.52}$$

In addition to $\zeta(3)$ in (2.34) one thus obtains similarly,

$$\zeta(5) = \frac{\pi^4}{3}\int_0^{\infty} ds\, s^{-6}\Big[2s^3\coth(s)[\sinh(s)]^{-2} + 3s^2[\sinh(s)]^{-2}$$
$$+ 3s\coth(s) - 8\Big], \tag{2.53}$$

$$\zeta(7) = \frac{\pi^6}{15}\int_0^{\infty} ds\, s^{-8}\Big[4s^4[\coth(s)]^2[\sinh(s)]^{-2}$$
$$+ 2s^4[\sinh(s)]^{-4} + 12s^3\coth(s)[\sinh(s)]^{-2}$$
$$+ 15s^2[\sinh(s)]^{-2} + 15s\coth(s) - 48\Big], \tag{2.54}$$

etc.

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[1]The condition $\mathrm{Re}(2n+2z) > 1$ takes effect only for $n = 0$, that is, we assume $(1/2) < \mathrm{Re}(z) < 1$ if $n = 0$.

[2]The second formula is mentioned since it appears to be advantageous (cf. (2.32)–(2.34)) to substitute $t = s^2$ after one performs the n differentiations w.r.t. t in the 1st line of (2.52).

Proof Assume that $n \in \mathbb{N}_0$, $0 < \operatorname{Re}(z) < 1$, and $\operatorname{Re}(2n + 2z) > 1$. Then,

$$\begin{aligned}
&\int_0^\infty dt\, t^{-z} \frac{d^n}{dt^n}\Big[\operatorname{tr}_{L^2((0,1);dx)} \big((-\Delta_D + t I_{L^2((0,1);dx)})^{-1}\big)\Big] \\
&\quad = \int_0^\infty dt\, t^{-z} \frac{d^n}{dt^n}\Big[(2t)^{-1}\big[t^{1/2}\coth\big(t^{1/2}\big) - 1\big]\Big] \\
&\quad = \sum_{k\in\mathbb{N}} \int_0^\infty dt\, t^{-z} \frac{d^n}{dt^n}\Big[\big(\pi^2 k^2 + t\big)^{-1}\Big] \\
&\quad = \sum_{k\in\mathbb{N}} \int_0^\infty dt\, \frac{t^{-z}(-1)^n n!}{\big(\pi^2 k^2 + t\big)^{n+1}} \\
&\quad = (-1)^n n! \sum_{k\in\mathbb{N}} \big(\pi^2 k^2\big)^{-z-n} \int_0^\infty du\, \frac{u^{-z}}{(1+u)^{n+1}} \\
&\quad = (-1)^n n! \pi^{-2(n+z)} \zeta(2(n+z)) \frac{\Gamma(1-z)\Gamma(n+z)}{\Gamma(n+1)} \\
&\quad = (-1)^n \pi^{-2(n+z)} \zeta(2(n+z)) \Gamma(1-z)\Gamma(n+z),
\end{aligned} \tag{2.55}$$

resulting in (2.52). (The condition $(1/2) < \operatorname{Re}(z) < 1$ if $n = 0$ guarantees convergence of the sum over k in (2.55).)

Alternatively, one can attempt to analytically continue the equation

$$\zeta(z) = \sin(\pi z/2)\pi^{z-1} \int_0^\infty ds\, s^{-z-1}[s\coth(s) - 1], \quad \operatorname{Re}(z) \in (1,2), \tag{2.56}$$

to the region $\operatorname{Re}(z) \geq 2$. For this purpose we first introduce

$$\coth(z) = \frac{1}{z} + \sum_{k=1}^\infty \frac{2^{2k} B_{2k}}{(2k)!} z^{2k-1}, \quad z \in \mathbb{C},\ 0 < |z| < \pi, \tag{2.57}$$

$$\begin{aligned}
&\coth_0(z) = \frac{1}{z}, \quad z \in \mathbb{C}\backslash\{0\}, \\
&\coth_n(z) = \frac{1}{z} + \sum_{k=1}^n \frac{2^{2k} B_{2k}}{(2k)!} z^{2k-1}, \quad z \in \mathbb{C}\backslash\{0\},\ n \in \mathbb{N}.
\end{aligned} \tag{2.58}$$

Theorem 2.59 *Let $n \in \mathbb{N}_0$, then,*

$$\begin{aligned}
\zeta(z) = \sin(\pi z/2)\pi^{z-1} \int_0^\infty ds\, s^{-z}[\coth(s) - \coth_n(s)], \\
\operatorname{Re}(z) \in (\max(1, 2n), 2n+2).
\end{aligned} \tag{2.60}$$

In particular,

$$\begin{aligned}\zeta(3) &= -\pi^2 \int_0^\infty ds\, s^{-3}[\coth(s) - \coth_1(s)] \\ &= -\pi^2 \int_0^\infty ds\, s^{-3}[\coth(s) - (1/s) - (s/3)],\end{aligned} \tag{2.61}$$

$$\begin{aligned}\zeta(5) &= \pi^4 \int_0^\infty ds\, s^{-5}[\coth(s) - \coth_2(s)] \\ &= \pi^4 \int_0^\infty ds\, s^{-5}\big[\coth(s) - (1/s) - (s/3) + \big(s^3/45\big)\big],\end{aligned} \tag{2.62}$$

$$\begin{aligned}\zeta(7) &= -\pi^6 \int_0^\infty ds\, s^{-7}[\coth(s) - \coth_3(s)] \\ &= -\pi^6 \int_0^\infty ds\, s^{-7}\big[\coth(s) - (1/s) - (s/3) + \big(s^3/45\big) - \big(2s^5/945\big)\big], \\ &\quad \textit{etc.}\end{aligned} \tag{2.63}$$

Proof When trying to analytically continue (2.56) to the right, one notices that it is the small s-behavior of the integrand that invalidates this representation. We therefore split the integral at some point $a > 0$ and write

$$\begin{aligned}\zeta(z) = \sin(\pi z/2)\pi^{z-1} &\int_a^\infty ds\, s^{-z-1}[s\coth(s) - 1] \\ + \sin(\pi z/2)\pi^{z-1} &\int_0^a ds\, s^{-z-1}[s\coth(s) - 1],\end{aligned} \tag{2.64}$$

where the first integral is well-defined for $\mathrm{Re}(z) > 1$, and the second for $\mathrm{Re}(z) < 2$. In order to analytically continue the second integral to the right, one writes

$$\begin{aligned}\int_0^a ds\, s^{-z-1}[s\coth(s) - 1] &= \int_0^a ds\, s^{-z-1}\left[s\coth(s) - 1 - \sum_{k=1}^n \frac{2^{2k}B_{2k}}{(2k)!}s^{2k}\right] \\ &\quad + \int_0^a ds\, s^{-z-1}\sum_{k=1}^n \frac{2^{2k}B_{2k}}{(2k)!}s^{2k} \\ &= \int_0^a ds\, s^{-z-1}\left[s\coth(s) - 1 - \sum_{k=1}^n \frac{2^{2k}B_{2k}}{(2k)!}s^{2k}\right]\end{aligned}$$

$$+\sum_{k=1}^{n}\frac{2^{2k}B_{2k}}{(2k)!}\int_0^a ds\, s^{-z-1+2k}$$
$$=\int_0^a ds\, s^{-z-1}\biggl[s\coth(s)-1-\sum_{k=1}^{n}\frac{2^{2k}B_{2k}}{(2k)!}s^{2k}\biggr]+\sum_{k=1}^{n}\frac{2^{2k}B_{2k}}{(2k)!}\frac{a^{2k-z}}{2k-z}, \tag{2.65}$$

valid for $1 < \mathrm{Re}(z) < 2n+2$, $z \notin \{2\ell\}_{1\le\ell\le n}$.

In summary, up to this point we have shown that for $1 < \mathrm{Re}(z) < 2n+2$, $z \notin \{2\ell\}_{1\le\ell\le n}$, and for $a > 0$, one has

$$\zeta(z)=\sin(\pi z/2)\pi^{z-1}\biggl\{\int_a^\infty ds\, s^{-z-1}[s\coth(s)-1]$$
$$+\int_0^a ds\, s^{-z-1}\biggl[s\coth(s)-1-\sum_{k=1}^{n}\frac{2^{2k}B_{2k}}{(2k)!}s^{2k}\biggr]$$
$$+\sum_{k=1}^{n}\frac{2^{2k}B_{2k}}{(2k)!}\frac{a^{2k-z}}{2k-z}\biggr\}. \tag{2.66}$$

Restricting z to $\mathrm{Re}(z) \in (\max(1,2n), 2n+2)$ and performing the limit $a \to \infty$ in (2.66), observing that the first and third terms on the right-hand side of (2.66) vanish in the limit, proves (2.60).

One notes that for $z = 2n$ the first two lines in (2.66) as well as all terms $k \neq n$ vanish and one confirms Euler's celebrated formula

$$\zeta(2n)=\lim_{z\to 2n}\biggl[\sin(\pi z/2)\pi^{z-1}\frac{2^{2n}B_{2n}}{(2n)!}\frac{a^{2n-z}}{2n-z}\biggr]=\frac{2^{2n-1}|B_{2n}|\pi^{2n}}{(2n)!}, \quad n\in\mathbb{N}. \tag{2.67}$$

Finally, one can take these investigations one step further as follows. Introducing

$$F(z)=\ln\bigl(z^{-1/2}\sinh\bigl(z^{1/2}\bigr)\bigr)=\sum_{k=1}^{\infty}\frac{2^{2k}B_{2k}}{2k(2k)!}z^k, \quad z\in\mathbb{C},\ |z|<\pi, \tag{2.68}$$

$$F_n(z)=\sum_{k=1}^{n}\frac{2^{2k}B_{2k}}{2k(2k)!}z^k, \quad z\in\mathbb{C},\ n\in\mathbb{N}, \tag{2.69}$$

one can show the following result.

Theorem 2.70 *Let $n \in \mathbb{N}_0$, then,*

$$\zeta(z)=(z/2)\pi^{z-1}\sin(\pi z/2)\int_0^\infty dt\, t^{-z/2-1}[F(t)-F_n(t)], \quad \mathrm{Re}(z)\in(\max(1,2n),2n+2). \tag{2.71}$$

In particular,

$$\begin{aligned}\zeta(3) &= -3\pi^2 \int_0^\infty ds\, s^{-4}\big[F\big(s^2\big) - F_1\big(s^2\big)\big] \\ &= -3\pi^2 \int_0^\infty ds\, s^{-4}\big[\ln\big(s^{-1}\sinh(s)\big) - \big(s^2/6\big)\big],\end{aligned} \tag{2.72}$$

$$\begin{aligned}\zeta(5) &= 5\pi^4 \int_0^\infty ds\, s^{-6}\big[F\big(s^2\big) - F_2\big(s^2\big)\big] \\ &= 5\pi^4 \int_0^\infty ds\, s^{-6}\big[\ln\big(s^{-1}\sinh(s)\big) - \big(s^2/6\big) + \big(s^4/180\big)\big],\end{aligned} \tag{2.73}$$

$$\begin{aligned}\zeta(7) &= -7\pi^6 \int_0^\infty ds\, s^{-8}\big[F\big(s^2\big) - F_3\big(s^2\big)\big] \\ &= -7\pi^6 \int_0^\infty ds\, s^{-8}\big[\ln\big(s^{-1}\sinh(s)\big) - \big(s^2/6\big) + \big(s^4/180\big) - \big(s^6/2835\big)\big], \\ &\textit{etc.}\end{aligned} \tag{2.74}$$

Proof The computation

$$\begin{aligned}F'(t) - F_n'(t) &= \frac{1}{2t}\big[t^{1/2}\coth\big(t^{1/2}\big) - 1\big] - \frac{1}{2}\sum_{k=1}^{n} \frac{2^{2k}B_{2k}}{(2k)!}\, t^{k-1} \\ &= \frac{1}{2t}\left[t^{1/2}\coth\big(t^{1/2}\big) - \sum_{k=0}^{n} \frac{2^{2k}B_{2k}}{(2k)!}\, t^{k}\right] \\ &= \frac{1}{2t}\big[t^{1/2}\coth\big(t^{1/2}\big) - t^{1/2}\coth_n\big(t^{1/2}\big)\big], \quad t \geq 0,\end{aligned} \tag{2.75}$$

and (2.60) then show

$$\begin{aligned}\zeta(z) &= \sin(\pi z/2)\pi^{z-1} \int_0^\infty ds\, s^{-z}[\coth(s) - \coth_n(s)] \\ &= \sin(\pi z/2)\pi^{z-1} \int_0^\infty dt\, t^{-z/2}[F'(t) - F_n'(t)] \\ &= (z/2)\sin(\pi z/2)\pi^{z-1} \int_0^\infty dt\, t^{-z/2-1}[F(t) - F_n(t)], \\ &\qquad \mathrm{Re}(z) \in (\max(1, 2n), 2n+2),\end{aligned} \tag{2.76}$$

after an integration by parts.

Acknowledgements We are indebted to the anonymous referee for kindly bringing references [24] and [27] to our attention.

Klaus Kirsten was supported by the Baylor University Summer Sabbatical and Research Leave Program.

Appendix: Basic Formulas for the Riemann ζ-Function

We present a number of formulas for $\zeta(z)$ and special values of $\zeta(\,\cdot\,)$. It goes without saying that no such collection can ever attempt at any degree of completeness, and certainly our compilation of formulas is no exception in this context.

Definition

$$\zeta(z) = \sum_{k\in\mathbb{N}} k^{-z}, \quad z \in \mathbb{C},\ \mathrm{Re}(z) > 1 \tag{A.1}$$

$$= \left[1 - 2^{-z}\right]^{-1} \sum_{k\in\mathbb{N}_0} (2k+1)^{-z}, \quad \mathrm{Re}(z) > 1, \quad [23, \text{p. } 19] \tag{A.2}$$

$$= \left[1 - 2^{1-z}\right]^{-1} \sum_{k\in\mathbb{N}} (-1)^{k+1} k^{-z}, \quad \mathrm{Re}(z) > 0, \quad [23, \text{p. } 19]. \tag{A.3}$$

Functional Equation

$$\zeta(z) = 2^z \pi^{z-1} \sin(\pi z/2) \Gamma(1-z) \zeta(1-z), \quad z \in \mathbb{C},\ \mathrm{Re}(z) < 0. \tag{A.4}$$

Alternative Formulas

$$\zeta(z) = \Gamma(z)^{-1} \int_0^\infty dt\, \frac{t^{z-1}}{e^t - 1}, \quad z \in \mathbb{C},\ \mathrm{Re}(z) > 1 \tag{A.5}$$

$$= \mu^z \Gamma(z)^{-1} \int_0^\infty dt\, \frac{t^{z-1}}{e^{\mu t} - 1}, \quad z \in \mathbb{C},\ \mathrm{Re}(z) > 1\ \mathrm{Re}(\mu) > 0, \quad [14, 3.4111] \tag{A.6}$$

$$= \Gamma(z)^{-1} [1 - 2^{1-z}]^{-1} \int_0^\infty dt\, \frac{t^{z-1}}{e^t + 1}, \quad z \in \mathbb{C},\ \mathrm{Re}(z) > 0 \tag{A.7}$$

$$= \mu^z \Gamma(z)^{-1} [1 - 2^{1-z}]^{-1} \int_0^\infty dt\, \frac{t^{z-1}}{e^{\mu t} + 1}, \quad z \in \mathbb{C},\ \mathrm{Re}(z) > 0\ \mathrm{Re}(\mu) > 0, \quad [14, 3.4113], \tag{A.8}$$

where

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}, \quad z \in \mathbb{C},\ \mathrm{Re}(z) > 0. \tag{A.9}$$

In addition,

$$\zeta(x) = \Gamma(x)^{-1} \int_0^1 \int_0^1 ds dt\, \frac{[\ln(st)]^{x-2}}{1 - st}, \quad x > 3, \quad [34] \tag{A.10}$$

$$= e^{i\pi(1-x)} \Gamma(x)^{-1} \int_0^1 dt\, \frac{\ln(t)^{x-1}}{1 - t}, \quad x > 1, \ \text{Jensen (1895)},\ [14, 4.2714], \tag{A.11}$$

$$= \pi^{z/2} \Gamma(z/2)^{-1} \int_0^\infty dt\, t^{(z/2)-1} \sum_{k \in \mathbb{N}} e^{-k^2 \pi t}, \tag{A.12}$$

and

$$\zeta(z) = \pi^{z/2} \Gamma(z/2)^{-1} \sum_{k \in \mathbb{N}} \int_0^\infty dt\, t^{(z/2)-1} e^{-k^2 \pi t}, \quad z \in \mathbb{C},\ \mathrm{Re}(z) > 1, \quad [32] \tag{A.13}$$

$$= \frac{2^{z-1}}{z-1} - 2^z \int_0^\infty dt\, \frac{\sin(z \arctan(t))}{(1+t^2)^{z/2}(e^{\pi t} + 1)}, \quad z \in \mathbb{C} \backslash \{1\}, \tag{A.14}$$

[14, 9.5134], [23, p. 21]

$$= \frac{2^{z-1}}{\left[1 - 2^{1-z}\right]} \int_0^\infty dt\, \frac{\cos(z \arctan(t))}{(1+t^2)^{z/2} \cosh(\pi t/2)}, \quad z \in \mathbb{C} \backslash \{1\}, \tag{A.15}$$

[23, p. 21]

$$= \frac{1}{2} + \frac{1}{z-1} + 2 \int_0^\infty dt\, \frac{\sin(z \arctan(t))}{(1+t^2)^{z/2}(e^{2\pi t} - 1)}, \quad z \in \mathbb{C} \backslash \{1\}, \tag{A.16}$$

Jensen's formula (1895), [23, p. 21]

$$= a^z \frac{2^{z-1}}{\left[2^z - 1\right]} \Gamma(z)^{-1} \int_0^\infty dt\, \frac{t^{z-1}}{\sinh(at)}, \quad \mathrm{Re}(z) > 1,\ a > 0,\ [14, 3.5231] \tag{A.17}$$

$$= \Gamma(z+1)^{-1} 4^{-1} (2a)^{z+1} \int_0^\infty dt\, \frac{t^z}{[\sinh(at)]^2}, \quad \mathrm{Re}(z) > -1,\ \mathrm{Re}(a) > 0, \tag{A.18}$$

[14, 3.5271]

$$= \Gamma(z+1)^{-1} 4^{-1} (2a)^{z+1} \big[1 - 2^{1-z}\big]^{-1} \int_0^\infty dt\, \frac{t^z}{[\cosh(at)]^2}, \tag{A.19}$$

$\mathrm{Re}(z) > -1,\ z \neq 1,\ \mathrm{Re}(a) > 0,$ [14, 3.5273]

$$= \Gamma(z+1)^{-1} \big[2 - 2^{2-z}\big]^{-1} \int_0^\infty dt\, \frac{t^z}{\cosh(t)+1}, \quad \mathrm{Re}(z) > 0,\ z \neq 1, \tag{A.20}$$

[14, 3.5316]

$$= 2^{-1} + \Gamma(z)^{-1} 2^{z-1} \int_0^\infty dt\, t^{z-1} e^{-2t} \coth(t), \quad \mathrm{Re}(z) > 1, \quad [14, 3.5513] \tag{A.21}$$

$$= 2^{-1} + \Gamma(z)^{-1} 2^{z-1} \int_0^\infty dt\, t^{z-1} e^{-2t} \coth(t), \quad \mathrm{Re}(z) > 1, \quad [14, 3.5513] \tag{A.22}$$

$$= \Gamma(z)^{-1} 2^{z-1} \int_0^\infty dt\, t^{z-1} \frac{e^{-t}}{\sinh(t)}, \quad \mathrm{Re}(z) > 1, \quad [14, 3.5521] \tag{A.23}$$

$$= \Gamma(z)^{-1} 2^{z-1} \big[1 - 2^{1-z}\big]^{-1} \int_0^\infty dt\, t^{z-1} \frac{e^{-t}}{\cosh(t)}, \quad \mathrm{Re}(z) > 0,\ z \neq 1,$$

$$[14, 3.5523] \tag{A.24}$$

$$= 2^z \Gamma(z)^{-1} \int_0^1 dt\, [\ln(1/t)]^{z-1} \frac{t}{1-t^2}, \quad \mathrm{Re}(z) > 0, \quad [14, 4.27212] \tag{A.25}$$

$$= \Gamma(z+1)^{-1} \int_0^\infty dt\, \frac{t^z e^t}{\big[e^t - 1\big]^2}, \quad \mathrm{Re}(z) > 1, \quad [23, \text{p. } 20] \tag{A.26}$$

$$= \Gamma(z+1)^{-1} \big[1 - 2^{1-z}\big]^{-1} \int_0^\infty dt\, \frac{t^z e^t}{\big[e^t + 1\big]^2}, \quad \mathrm{Re}(z) > 0, \quad [23, \text{p. } 20] \tag{A.27}$$

$$= 2 \sin(\pi z/2) \int_0^\infty dt\, \frac{t^{-z}}{e^{2\pi t} - 1}, \quad \mathrm{Re}(z) < 0, \quad [22, \text{p. } 104] \tag{A.28}$$

$$= (2^z - 1)^{-1} \frac{2^{z-1} z}{z-1} + 2(2^z - 1)^{-1} \int_0^\infty dt\, \frac{\sin(z \arctan(2t))}{[(1/4) + t^2]^{z/2}} \frac{1}{e^{2\pi t} - 1}, \tag{A.29}$$

$z \in \mathbb{C}\backslash\{1\},$ [30, p. 279].

Specific Values

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n} B_{2n}}{2(2n)!}, \quad n \in \mathbb{N}_0, \tag{A.30}$$

where B_m are the Bernoulli numbers generated, for instance, by

$$\frac{w}{e^w - 1} = \sum_{m \in \mathbb{N}_0} B_m \frac{w^m}{m!}, \quad w \in \mathbb{C},\ |w| < 2\pi, \tag{A.31}$$

in particular,

$$B_0 = 1,\ B_1 = -1/2,\ B_2 = 1/6,\ B_3 = 0,\ B_4 = -1/30,\ B_5 = 0,\ B_6 = 1/42, \text{etc.}, \tag{A.32}$$

$$B_{2k+1} = 0,\ k \in \mathbb{N}. \tag{A.33}$$

Moreover, one has the **generating functions** for $\zeta(2n)$,

$$-(\pi z/2)\cot(\pi z) = \sum_{n \in \mathbb{N}_0} \zeta(2n) z^{2n}, \quad |z| < 1,\ \zeta(0) = -1/2, \tag{A.34}$$

$$-(\pi z/2)\coth(\pi z) = \sum_{n \in \mathbb{N}_0} (-1)^n \zeta(2n) z^{2n}, \quad |z| < 1,\ \zeta(0) = -1/2, \tag{A.35}$$

and [32]

$$(n!/6)[\zeta(n-2) - 3\zeta(n-1) + 2\zeta(n)] = \int_0^\infty dt\, \frac{t^n e^t}{(e^t - 1)^4}, \quad n \in \mathbb{N},\ n \geq 4. \tag{A.36}$$

Choosing $k = 2n$, $n \in \mathbb{N}$, even, employing (A.30) for $\zeta(2n)$, $\zeta(2n-2)$, yields a formula for $\zeta(2n-1)$. Moreover,

$$\zeta(2n+1) = \frac{1}{(2n)!} \int_0^\infty dt\, \frac{t^{2n}}{e^t - 1}, \quad n \in \mathbb{N} \tag{A.37}$$

$$= \frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!} \int_0^1 dt\, B_{2n+1}(t)\cot(\pi t), \quad n \in \mathbb{N}, \quad [9], \tag{A.38}$$

where $B_m(\,\cdot\,)$ are the Bernoulli polynomials,

$$B_m(z) = \sum_{j=0}^{m} \binom{m}{j} B_j z^{m-j}, \quad t \in \mathbb{C}, \tag{A.39}$$

generated, for instance, by

$$\frac{we^{zw}}{e^w - 1} = \sum_{m\in\mathbb{N}_0} B_m(z)\frac{w^m}{m!}, \quad w \in \mathbb{C},\ |w| < 2\pi. \tag{A.40}$$

Explicitly,

$$\begin{aligned} &B_0(x) = 1,\ B_1(x) = x - (1/2),\ B_2(x) = x^2 - x + (1/6), \\ &B_3(x) = x^3 - (3/2)x^2 + (1/2)x,\ \text{etc.}, \end{aligned} \tag{A.41}$$

$$B_n(0) = B_n,\ n \in \mathbb{N}, \quad B_1(1) = -B_1 = 1/2,\ B_n(1) = B_n,\ n \in \mathbb{N}_0\backslash\{1\}, \tag{A.42}$$

$$B_n'(x) = nB_{n-1}(x), \quad n \in \mathbb{N},\ x \in \mathbb{R}. \tag{A.43}$$

In addition, for $n \in \mathbb{N}$,

$$\begin{aligned} \zeta(2n+1) = {} & \frac{a^2(2a)^{2n}}{[2^{-2n-1} - 1]}\frac{1}{(2n+1)!} \\ & \times \int_0^\infty dt\, t^{2n+1}\frac{\cosh(at)}{[\sinh(at)]^2}, \quad a \neq 0, \quad [14, 3.5279] \end{aligned} \tag{A.44}$$

$$= \frac{2^{2n}}{\left[2^{2n} - 1\right]}[(2n)!]^{-1}\int_0^1 dt\, \frac{[\ln(t)]^{2n}}{1+t}, \quad [14, 4.2711] \tag{A.45}$$

$$= \frac{2^{2n+1}}{\left[2^{2n+1} - 1\right]}[(2n)!]^{-1}\int_0^1 dt\, \frac{[\ln(t)]^{2n}}{1-t^2}, \quad [14, 4.2711], \tag{A.46}$$

$$\zeta(n) = [(n-1)!]^{-1}\int_0^1 dt\, \frac{[\ln(1/t)]^{n-1}}{1-t}, \quad [14, 4.2729]. \tag{A.47}$$

Just for curiosity,

$$\zeta(3) = 1.2020569032\ldots.. \tag{A.48}$$

Apery [1] proved in 1978 that $\zeta(3)$ is irrational (see also Beukers [2], van der Poorten [29], Zudilin [35], and [31], [33]).

Moreover,

$$\zeta(3) = \sum_{k\in\mathbb{N}} k^{-3} = \frac{8}{7}\sum_{k\in\mathbb{N}_0}(2k+1)^{-3} = \frac{4}{3}\sum_{k\in\mathbb{N}_0}(-1)^k(k+1)^{-3}, \quad [31] \tag{A.49}$$

$$= \frac{1}{2}\int_0^\infty dt\, \frac{t^2}{e^t - 1}, \quad [31] \tag{A.50}$$

$$= \frac{2}{3}\int_0^\infty dt\,\frac{t^2}{e^t+1}, \quad [31] \tag{A.51}$$

$$= \frac{4}{7}\int_0^{\pi/2} dt\,t\ln([1/\cos(t)]+\tan(t)), \quad [31] \tag{A.52}$$

$$= \frac{8}{7}\left[\frac{\pi^2\ln(2)}{4} + 2\int_0^{\pi/2} dt\,t\ln(\sin(t))\right], \quad [33] \tag{A.53}$$

$$= -\frac{1}{2}\int_0^1\int_0^1 dxdy\,\frac{\ln(xy)}{1-xy}, \quad [2] \tag{A.54}$$

$$= \int_0^1\int_0^1\int_0^1 dxdydz\,\frac{1}{1-xyz}, \quad [31] \tag{A.55}$$

$$= \pi\int_0^\infty dt\,\frac{\cos(2\arctan(t))}{(1+t^2)[\cosh(\pi t/2)]^2}, \quad [31] \tag{A.56}$$

$$= \frac{8\pi^2}{7}\int_0^1 dt\,\frac{t(t^4-4t^2+1)\ln(\ln(1/t))}{(1+t^2)^4}, \quad [31] \tag{A.57}$$

$$= \frac{8\pi^2}{7}\int_1^\infty dt\,\frac{t(t^4-4t^2+1)\ln(\ln(t))}{(1+t^2)^4}, \quad [31] \tag{A.58}$$

$$= 10\int_0^{1/2} dt\,\frac{[\mathrm{arcsinh}(t)]^2}{t} \quad [12, \text{p. } 46] \tag{A.59}$$

$$= (2/7)\pi^2\ln(2) + (4/7)\int_0^\pi dt\,t\ln(\sin(t/2)) \quad [12, \text{p. } 46] \tag{A.60}$$

$$= (2/7)\pi^2\ln(2) - (8/7)\int_0^1 dt\,\frac{[\arcsin(t)]^2}{t} \quad [12, \text{p. } 46] \tag{A.61}$$

$$= (2/7)\pi^2\ln(2) - (8/7)\int_0^{\pi/2} dt\,t^2\cot(t) \quad [12, \text{p. } 46] \tag{A.62}$$

$$\zeta(3) = -\frac{2}{7}\pi^2\ln(2) - \frac{16}{7}\int_0^1 dt\,\frac{\mathrm{arctanh}(t)\ln(t)}{t(1-t^2)} \tag{A.63}$$

$$= -\frac{4}{3}\int_0^1 dt\,\frac{\ln(t)\ln(1+t)}{t} \tag{A.64}$$

$$= -8\int_0^1 dt\,\frac{\ln(t)\ln(1+t)}{1+t} \tag{A.65}$$

$$= \int_0^1 dt\,\frac{\ln(t)\ln(1-t)}{1-t} = \int_0^1 dt\,\frac{\ln(t)\ln(1-t)}{t} \tag{A.66}$$

$$= \frac{1}{4}\pi^2 \ln(2) + \int_0^1 dt\, \frac{\ln(t)\ln(1+t)}{1-t} \tag{A.67}$$

$$= \frac{2}{13}\pi^2 \ln(2) + \frac{8}{13} \int_0^1 dt\, \frac{\ln(t)\ln(1-t)}{1+t} \tag{A.68}$$

$$= \frac{2}{7} \int_0^{\pi/2} dt\, \frac{t(\pi - t)}{\sin(t)}. \tag{A.69}$$

Formulas (A.63)–(A.69) were provided by Glasser and Ruehr and can be found in [21, Problem 80-13]. Finally, we also recall,

$$\zeta(3) = 1 + \int_0^\infty dt\, \frac{6t - 2t^3}{(1+t^2)^3}\, \frac{1}{e^{2\pi t} - 1}, \quad [17, \text{p. } 274] \tag{A.70}$$

$$= \frac{6}{7} + \frac{2}{7} \int_0^\infty dt\, \frac{\sin(3\arctan(2t))}{[(1/4)+t^2]^{3/2}}\, \frac{1}{e^{2\pi t} - 1} \tag{A.71}$$

$$= \frac{6}{7} + \frac{8}{7} \int_0^\infty dt\, \frac{\sin(3\arctan(t))}{(1+t^2)^{3/2}}\, \frac{1}{e^{\pi t} - 1} \tag{A.72}$$

$$= 2 - 8 \int_0^\infty dt\, \frac{\sin(3\arctan(t))}{(1+t^2)^{3/2}}\, \frac{1}{e^{\pi t} + 1} \tag{A.73}$$

$$= 1 + 2 \int_0^\infty dt\, \frac{\sin(3\arctan(t))}{(1+t^2)^{3/2}}\, \frac{1}{e^{2\pi t} - 1}. \tag{A.74}$$

Formulas (A.71)–(A.73) are due to Jensen (1895) and are special cases of results to be found in [30, p. 279] (cf. (A.16), (A.29)); finally, (A.74) is a consequence of (A.72) and (A.73).

For more on $\zeta(3)$ see also [12, p. 42–45].

For a wealth of additional formulas, going beyond what is recorded in this appendix, we also refer to [24] and [27].

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