Green's Functions and Euler's Formula for $\zeta(2n)$

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Abstract In this note, we calculate the Green's function for the linear operator $(-\Delta_D)^n$, where $-\Delta_D$ is the one-dimensional Dirichlet Laplacian in $L^2((0,1);dx)$ defined by

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$$-\Delta_D f = -f''$$

with (Dirichlet) boundary conditions $f(0) = f(1) = 0$. As a consequence of this computation, we obtain Euler's formula

$$\zeta(2n) = \sum_{k\in\mathbb{N}} k^{-2n} = \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}, \quad n \in \mathbb{N},$$

where $\zeta(\cdot)$ denotes the Riemann zeta function and B_n is the nth Bernoulli number. This generalizes the example given by Grieser [29] for $n = 1$. In addition, we derive its z-dependent generalization for $z \in \mathbb{C}\backslash\big\{(k\pi)^{2n}\big\}_{k\in\mathbb{N}}$,

$$\sum_{k\in\mathbb{N}} \big[(k\pi)^{2n} - z\big]^{-1} = \frac{1}{2nz}\Bigg[n - \sum_{j=0}^{n-1} \omega_j^{1/2} z^{1/(2n)} \cot\big(\omega_j^{1/2} z^{1/(2n)}\big)\Bigg], \quad n \in \mathbb{N},$$

where $\omega_j = e^{2\pi i j/n}$, $0 \le j \le n-1$, represent the nth roots of unity. In this context we also derive the Green's function of $\big((-\Delta_D)^n - zI\big)^{-1}$, $n \in \mathbb{N}$.

Keywords Dirichlet Laplacian · Green's function · Trace class operators · Trace formulas · Riemann zeta function · Bernoulli numbers · Bernoulli polynomials

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1 Introduction

An astute student in a sophomore differential equations course can compute the set of eigenvalues of the Dirichlet boundary value problem

$$-\Delta_D f = -f'', \quad f(0) = f(1) = 0. \tag{1.1}$$

Indeed, the underlying linear operator $-\Delta_D$ is the positive, self-adjoint operator in the Hilbert space $L^2((0, 1); dx)$,

$$\begin{aligned} &(-\Delta_D f)(x) = -f''(x) \text{ for a.e. } x \in (0, 1),\\ &f \in \operatorname{dom}(-\Delta_D) = \big\{g \in L^2((0, 1); dx) \,\big|\, g, g' \in AC([0, 1]);\ g(0) = g(1) = 0;\\ &\qquad g'' \in L^2((0, 1); dx)\big\} \end{aligned} \tag{1.2}$$

($AC([0, 1])$ denotes the set of absolutely continuous functions on $[0, 1]$), with purely discrete spectrum,

$$\sigma(-\Delta_D) = \{\lambda_k = (k\pi)^2\}_{k\in\mathbb{N}}. \tag{1.3}$$

The eigenspace of each λ_k is one-dimensional and spanned by the normalized eigenfunctions

$$\begin{aligned} &-\Delta_D u_k = (k\pi)^2 u_k, \quad u_k(x) = 2^{1/2}\sin(k\pi x),\ 0 \le x \le 1, \\ &\|u_k\|_{L^2((0,1);dx))} = 1, \quad k \in \mathbb{N}. \end{aligned} \tag{1.4}$$

In particular, the collection $\{2^{1/2}\sin(k\pi x)\}_{k\in\mathbb{N}}$ forms a complete orthonormal basis in $L^2((0,1);dx)$. Since $0 \notin \sigma(-\Delta_D)$, $(-\Delta_D)^{-1}$ exists and is explicitly given by

$$\big((-\Delta_D)^{-1} f\big)(x) = \int_0^1 dy\, K_1(x,y) f(y), \quad f \in L^2((0,1);dx), \tag{1.5}$$

where $K_1(\,\cdot\,,\,\cdot\,)$ denotes the Green's function for $-\Delta_D$ given by

$$K_1(x,y) = \begin{cases} x(1-y), & 0 \le x \le y \le 1, \\ y(1-x), & 0 \le y < x \le 1. \end{cases} \tag{1.6}$$

This operator $(-\Delta_D)^{-1}$ is a bounded, self-adjoint, compact operator with eigenvalues $\{\lambda_k^{-1}\}_{k=1}^{\infty}$ and associated eigenfunctions $\{u_k\}_{k=1}^{\infty}$. Moreover, as discussed below, $(-\Delta_D)^{-1}$ is a trace class operator and this implies

$$\sum_{k\in\mathbb{N}} \lambda_k^{-1} = \int_0^1 dx\, K_1(x,x) = \frac{1}{6} \tag{1.7}$$

from which the solution to the famous "*Basel problem*" quickly emerges:

$$\sum_{k\in\mathbb{N}} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{1.8}$$

This example, and methodology, was studied by Grieser [29] in a paper centered on trace formulas. In this note, we generalize Grieser's work by explicitly computing the Green's function $K_n(\cdot,\cdot)$ associated with $(-\Delta_D)^n$ for any integer $n \ge 1$ and from this we deduce, as a corollary, Euler's formula for $\zeta(2n)$:

$$\zeta(2n) = \sum_{k\in\mathbb{N}} \frac{1}{k^{2n}} = \frac{(-1)^{n-1} 2^{2n-1} \pi^{2n} B_{2n}}{(2n)!}, \quad n \in \mathbb{N}; \tag{1.9}$$

here $\zeta(\,\cdot\,)$ denotes the Riemann zeta function and $\{B_n\}_{n\in\mathbb{N}_0}$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) denotes the sequence of Bernoulli numbers. In our final section we derive a z-dependent generalization of (1.9) for $z \in \mathbb{C}\backslash\{(k\pi)^{2n}\}_{k\in\mathbb{N}}$,

$$\sum_{k\in\mathbb{N}}\big[(k\pi)^{2n}-z\big]^{-1}=\frac{1}{2nz}\left[n-\sum_{j=0}^{n-1}\omega_j^{1/2}z^{1/(2n)}\cot\left(\omega_j^{1/2}z^{1/(2n)}\right)\right],\quad n\in\mathbb{N}, \tag{1.10}$$

where $\omega_j=e^{2\pi ij/n}$, $0\le j\le n-1$, represent the nth roots of unity. To derive (1.10) we explicitly compute the Green's function $K_n(z;\,\cdot\,,\,\cdot\,)$ of $\big((-\Delta_D)^n-zI\big)^{-1}$, $n\in\mathbb{N}$.

There are several excellent expository articles in the literature concerning the history of the zeta function and, more specifically, Euler's formula (1.9). The paper by Ayoub [6] on the zeta function includes discussion of some of Euler's original proofs of the computation of $\zeta(2)$ (the Basel problem). Kline [39] gives a thorough account of Euler's early influence on infinite series, including Euler's evaluation of $\zeta(2n)$ for $n\in\mathbb{N}$. Two important sources on the calculation of $\zeta(2)$ are the article [53] and book [54, Chap. 21] by Sandifer who details three of Euler's solutions of the Basel problem beginning in 1735. We remark that the book by Roy [52] notes that Euler gave *eight* solutions of the Basel problem in his career. As noted in [7], the Basel problem was open for 91 years before Euler supplied the first known proof. Indeed, Pietro Mengoli posed this problem in 1644 (in 1655, John Wallis commented on this problem in his book *Arithmetica Infinitorum*).

Behind the scenes of many known proofs of (1.9) are methods relying on Cauchy's residue calculus, Weierstrass' product theorem, Parseval's theorem, and Fourier expansions. There are at least three 'most common' proofs of (1.9) given in various texts. One of them is the original proof given by Euler [24] (see also [40] [Chap. VI, Sect. 24]) in 1741. His proof of (1.9) involved matching coefficients for two different representations of $\pi z\cot(\pi z)$, one a power series expansion and the other a partial fractions decomposition; see [3] for a short, but concise, explanation of Euler's method.

A second well-known proof of (1.9) is obtained by setting $z=2n$ in Riemann's functional equation

$$\zeta(1-z)=2(2\pi)^{-z}\Gamma(z)\cos(\pi z/2)\zeta(z) \tag{1.11}$$

and using the fact that $\zeta(1-2n)=-B_{2n}/(2n)$; see [1] [23.2.6 and 23.2.15].

A third standard proof is to evaluate the Fourier cosine series of $B_{2n}(x)$, the Bernoulli polynomial of degree $2n$, at $x=0$; for example, see [1] [23.1.18] or [4] [Theorem 12.19].

There are several additional contributions in the literature regarding Euler's formula (1.9). Some of these references deal with new proofs for the special case $n=1$, the Basel problem case (which we call group 1) and other proofs discuss the more general $n\in\mathbb{N}$ case (group 2). We briefly discuss the contents of both groups.

For group 1, we note that, in his book [22] [Chap. 3], Dunham illustrates, in detail, one of Euler's solutions by computing $\zeta(2)$ using the infinite product expansion of $\sin(x)/x$, namely,

$$\frac{\sin(x)}{x}=\prod_{k\in\mathbb{N}}\left(1-\frac{x^2}{k^2\pi^2}\right). \tag{1.12}$$

Specifically, using the Maclaurin series for $\sin(x)$, expanding the right-hand side of (1.12) into a Maclaurin series and then comparing coefficients of x^2 on both sides, the value of $\zeta(2)$ is found. Another proof of the computation of $\zeta(2)$ was given by Papadimitriou in [48]; his remarkable method stems from the simple inequality $\sin(x) < x < \tan(x)$ on $(0, \pi/2)$; see also [34] and [57] which both use ideas similar to those used by Papadimitriou. Choe [12] computes $\zeta(2)$ by first cleverly showing that

$$\sum_{k\in\mathbb{N}_0} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8} \tag{1.13}$$

and noting, by absolute convergence, that

$$\zeta(2) = \sum_{k\in\mathbb{N}_0} \frac{1}{(2k+1)^2} + \sum_{k\in\mathbb{N}}^{\infty} \frac{1}{(2k)^2} = \frac{4}{3}\sum_{k\in\mathbb{N}_0} \frac{1}{(2k+1)^2}; \tag{1.14}$$

(it should be noted that Euler also used this 'trick' to compute $\zeta(2)$ in one of his solutions). Kimble [38] offers a variation of this proof by employing the Maclaurin series of $\arcsin(x)$ and using the identity

$$\frac{\pi^2}{6} = \frac{4}{3}\frac{(\arcsin(1))^2}{2} = \frac{4}{3}\int_0^1 dx\, \frac{\arcsin(x)}{\sqrt{1-x^2}}. \tag{1.15}$$

In [5], Apostol gives another proof of Euler's formula for $n = 1$ by evaluating the double integral

$$\int_0^1 \int_0^1 dxdy \frac{1}{1-xy}. \tag{1.16}$$

In [59, 62], Stark also supplies new proofs of $\zeta(2)$. Likewise, the following contributions all deal with interesting techniques, some new and some variations of older methods, of computing $\zeta(2)$: Benko [7], Benko and Molokach [8], Chapman [10], Daners [17], Giesey [26], Harper [31], Hirschhorn [32], Hofbauer [33], Ivan [36], Knopp and Schur [41], Kortram [42], Marshall [44], Matsuoka [45], Passare [49], and Vermeeren [67].

We comment briefly on the contents of group two, the set of papers dealing with proofs of the computation of $\zeta(2n)$ for each $n \in \mathbb{N}$. Apostol [3] generalizes Papadimitriou's method ($n = 1$) to compute $\zeta(2n)$. Two different proofs of (1.9) were given by Berndt in [9]; one proof uses a special case of the Riemann–Lebesgue lemma and his second proof is, essentially, a further generalization of Apostol's proof in [3]. More generally, the identity

$$\sum_{k\in\mathbb{N}} \frac{1}{k^2 - z} = \frac{1 - \pi z^{1/2}\cot\left(\pi z^{1/2}\right)}{2z}, \quad z \in \mathbb{C}\backslash\mathbb{N}. \tag{1.17}$$

already appeared in Titchmarsh [65] [p. 113]; it was later employed in Dikii [21] [Eq. (3.9)] in his seminal paper on (regularized) trace formulas for Sturm–Liouville operators and eigenvalue asymptotics. In a recent text, Teschl [64] [p. 85] uses trace methods to derive (1.17), and, by comparing the power series of both sides of (1.17) at $z = 0$, he deduces (1.9). The recent contribution by Alladi and Defant [2] establishes (1.9) by applying Parseval's identity to the Fourier coefficients of the periodic function x^k and using induction on k; see also [43]. Other interesting, and diverse, proofs of (1.9) are given by Chen [11], Chungang and Yonggao [13], Ciaurri, Navas, Ruiz and Varona [14], de Amo, Díaz Carillo and Fernández-Sánchez [19], Estermann [23], Hovstad [35], Osler [47], Robbins [51], Stark [60, 61], Tsumura [66], Williams [69], and Williams [70]. For zeta functions associated to general Sturm–Liouville operators we refer to the classical work of Dikii [20, 21] (see also the recent article [25] and the extensive literature cited therein).

Lastly, we note that there are q version results of the identity in (1.9). Indeed, Goswami [30] recently gave a natural family of identities whose limits as $q \to 1$ with $|q| < 1$ give Euler's formula for each $n \in \mathbb{N}$; see also the contribution [63] by Sun.

2 Bernoulli Polynomials and Bernoulli Numbers

For properties of Bernoulli polynomials and Bernoulli numbers, we refer the reader to the standard sources [1] [Chap. 23] and [27] [Sects. 9.6 and 9.7] as well as the online Digital Library of Mathematical Functions https://dlmf.nist.gov/ and the accompanying book [46].

The Bernoulli polynomials $\{B_n(x)\}_{n\in\mathbb{N}_0}$ can be defined through the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n\in\mathbb{N}_0} B_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi, \; x \in \mathbb{R}. \tag{2.1}$$

For the record, the first few Bernoulli polynomials are given by

$$\begin{aligned} &B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ &B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \text{ etc.} \end{aligned} \tag{2.2}$$

Among several properties that these polynomials satisfy, we will make use of the fact that

$$\int_0^1 dx\, B_{2n}(x) = 0, \quad n \geq 1, \tag{2.3}$$

which follows from the identities (see [1][23.1.8 and 23.1.11])

$$B_n(1-x) = (-1)^n B_n(x), \quad n \geq 1, \tag{2.4}$$

and

$$\int_0^x du\, B_n(u) = \frac{B_{n+1}(x) - B_{n+1}(0)}{n+1}, \quad n \geq 1. \tag{2.5}$$

We also recall the Fourier cosine series expansion of $B_{2n}(x)$ (see [1] [23.1.18] or [4] [Theorem 12.19]), which is given by

$$\frac{(-1)^{n-1}(2n)!}{2^{2n-1}\pi^{2n}} \sum_{k\in\mathbb{N}} \frac{\cos(2k\pi x)}{k^{2n}} = B_{2n}(x), \quad 0 \leq x \leq 1, \ n \in \mathbb{N}. \tag{2.6}$$

The Bernoulli numbers $\{B_n\}_{n\in\mathbb{N}_0}$ are defined as $B_n = B_n(0)$. For example,

$$B_0 = 1, \ B_1 = -1/2, \ B_2 = 1/6, \ B_4 = -1/30, \ B_6 = 1/42, \ B_8 = -1/30, \\ B_{10} = 5/66, \ \text{ etc., } B_{2n+1} = 0 \ \text{ for } n \geq 1. \tag{2.7}$$

We will see in Sect. 4, that the Green's function $K_n(\cdot,\cdot)$, associated with the n^{th} power of the operator $-\Delta_D$ defined in (1.1), can be given explicitly in terms of Bernoulli polynomials.

3 On Trace Class Operators

A classical result in an elementary linear algebra course states that, for an arbitrary $n \times n$ matrix $B = (B_{j,k})_{1\leq j,k\leq n}$ of complex numbers with eigenvalues $\{\beta_k\}_{k=1}^n$ (counting algebraic multiplicities), one has

$$\operatorname{tr}(B) = \sum_{j=1}^n B_{j,j} = \sum_{k=1}^n \beta_k. \tag{3.1}$$

This result generalizes to an important class of compact operators in an infinite-dimensional, separable Hilbert space $(\mathcal{H}, (\,\cdot\,,\,\cdot\,)_{\mathcal{H}})$, the so-called trace class operators. Specifically, a bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is *trace class* if, for some (and hence for all) orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $\mathcal{H}$, the sum

$$\sum_{j\in\mathbb{N}} (e_j, (T^*T)^{1/2} e_j)_{\mathcal{H}} \tag{3.2}$$

is finite; see [56] [Sect. 3.6] for an in-depth discussion of this topic. In this case, the trace of T is defined to be

$$\operatorname{tr}(T) = \sum_{j\in\mathbb{N}} (e_j, Te_j)_{\mathcal{H}}; \tag{3.3}$$

this finite-valued sum is absolutely convergent and independent of the choice of orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$. In addition, if $\{\tau_k\}_{k\in J}$ (with $J \subseteq \mathbb{N}$ an appropriate index set) represent the eigenvalues of T, counting algebraic multiplicities, then by Lidskii's theorem (see, e.g., [55] [Chap. 3], [56] [Sect. 3.12])

$$\operatorname{tr}(T) = \sum_{j\in\mathbb{N}} (e_j, Te_j)_{\mathcal{H}} = \sum_{k\in J} \tau_k. \tag{3.4}$$

For the purpose of this note, we discuss a particular set of trace class integral operators. Let $[a, b] \subset \mathbb{R}$ be a compact interval and $K(\cdot\,,\,\cdot) : [a, b] \times [a, b] \to \mathbb{C}$ be a continuous function. Define $T : L^2((a, b); dx) \to L^2((a, b); dx)$ by

$$(Tf)(x) = \int_a^b dy\, K(x, y) f(y), \quad f \in L^2((a, b); dx), \tag{3.5}$$

and assume that $T \geq 0$ (implying $K(x, x) \geq 0$, $x \in [a, b]$). Then by Mercer's Theorem (see, e.g.., [18] [Proposition 5.6.9], [56] [Theorem 3.11.9]), T is trace class and

$$\operatorname{tr}(T) = \int_a^b dx\, K(x, x) \in [0, \infty). \tag{3.6}$$

We now see that the formulas given in (3.4) and (3.6) afford us two different means to compute the trace of an integral operator of the type defined in (3.5); this observation is fundamental in our pursuit of establishing Euler's formula (1.9).

We now connect these trace class results to the inverse of the self-adjoint operator $(-\Delta_D)^n$, where $n \in \mathbb{N}$ and $-\Delta_D$ is the second-order differential operator defined in (1.1). It is straightforward to see that the two-point boundary value problem defined by $(-\Delta_D)^n$ is given by

$$\begin{aligned} &\big((-\Delta_D)^n f\big)(x) = (-1)^n f^{(2n)}(x) \ \text{ for a.e. } x \in (0, 1), \\ &f \in \operatorname{dom}\big((-\Delta_D)^n\big) = \{g \in L^2((0, 1); dx) \,\big|\, g^{(k)} \in AC([0, 1]),\ 0 \leq k \leq 2n - 1; \\ &\qquad g^{(2j)}(0) = g^{(2j)}(1) = 0,\ 0 \leq j \leq n - 1;\ g^{(2n)} \in L^2((0, 1); dx)\}. \end{aligned} \tag{3.7}$$

The spectrum of $(-\Delta_D)^n$ is given by

$$\sigma\big((-\Delta_D)^n\big) = \big(\sigma\big(-\Delta_D\big)\big)^n = \big\{\lambda_k^n = (k\pi)^{2n}\big\}_{k\in\mathbb{N}}, \tag{3.8}$$

where λ_k was introduced in (1.3). Furthermore, since $0 \notin \sigma\big((-\Delta_D)^n\big)$ for all $n \in \mathbb{N}$, one infers that $\big((-\Delta_D)^n\big)^{-1} = (-\Delta_D)^{-n}$ exists with eigenvalues $\lambda_k^{-n} = (k\pi)^{-2n}$, $k \in$

$\mathbb{N}$, and corresponding eigenfunctions $\{u_k\}_{k=1}^{\infty}$ defined in (1.4). The inverse $(-\Delta_D)^{-n}$: $L^2((0,1);dx) \to L^2((0,1);dx)$ of $(-\Delta_D)^n$ is a bounded, self-adjoint, and compact integral operator given by

$$\big((-\Delta_D)^{-n} f\big)(x) = \int_0^1 dy\, K_n(x,y) f(y), \quad f \in L^2((0,1);dx), \tag{3.9}$$

where $K_n(\cdot,\cdot)$ is the unique Green's function for $(-\Delta_D)^n$ in the sense of [15] [Theorem 7.2.2]. In particular, the Green's function $K_n(\cdot,\cdot)$ has the following properties: (i) $K_n(\cdot,\cdot)$ is symmetric on $[0,1]\times[0,1]$ and for fixed $y \in [0,1]$, $K_n(\cdot,y)$ and its first $(2n-1)$ derivatives are continuous on $[0,y)$ and $(y,1]$. At the point $x=y$, $K_n(\cdot,\cdot)$ and its first $(2n-2)$ derivatives have removable singularities (i.e., the left and right limits exist and equal each other), while the $(2n-1)$st derivative has a jump discontinuity satisfying

$$\frac{\partial^{2n-1} K_n(y^+,y)}{\partial x^{2n-1}} - \frac{\partial^{2n-1} K_n(y^-,y)}{\partial x^{2n-1}} = (-1)^n. \tag{3.10}$$

(ii) As a function of $x \in [0,1]\backslash\{y\}$, $K_n(\,\cdot\,,y)$, is a solution of the differential equation

$$\big((-\Delta_D)^n f\big)(x) = 0, \quad x \in [0,1]\backslash\{y\}, \tag{3.11}$$

and as a function of $x \in [0,1]$, $K_n(\,\cdot\,,y)$, satisfies the boundary conditions

$$f^{(2j)}(0) = f^{(2j)}(1) = 0, \quad 0 \le j \le n-1, \tag{3.12}$$

for each $y \in [0,1]$.
(iii) For each $f \in L^2((0,1);dx)$, the solution $u \in \operatorname{dom}\big((-\Delta_D)^n\big)$ of $(-\Delta_D)^n u = f$ is of the form

$$u(x) = \int_0^1 dy\, K_n(x,y) f(y) \ \text{ for a.e. } x \in [0,1]. \tag{3.13}$$

4 The Green's Function for $(-\Delta_D)^n$, $n \in \mathbb{N}$

Following the general method developed, for example in [16] [Sect. 6.7] (see also [58]), some (increasingly) tedious computations yield the following explicit Green's functions $K_n(x,y)$ for $n = 2, 3$, and 4:

$$K_2(x,y) = \begin{cases} \frac{1}{6}x(y-1)(x^2+y^2-2y), \ 0 \le x \le y \le 1, \\ \frac{1}{6}y(x-1)(y^2+x^2-2x), \ 0 \le y < x \le 1, \end{cases} \tag{4.1}$$

$$K_3(x,y) = \begin{cases} \frac{1}{360}x(1-y)(8y-20x^2y+10x^2y^2+8y^2-12y^3+3y^4+3x^4), \\ \qquad 0 \le x \le y \le 1, \\ \frac{1}{360}y(1-x)(8x-20xy^2+10x^2y^2+8x^2-12x^3+3x^4+3y^4), \\ \qquad 0 \le y < x \le 1, \end{cases} \tag{4.2}$$

and

$$K_4(x,y) = \begin{cases} \frac{1}{15120}x(y-1)(-32y+56x^2y-42x^4y+56x^2y^2-84x^2y^3 \\ +21x^2y^4+21x^4y^2+3x^6-32y^2+24y^3+24y^4-18y^5+3y^6), \\ \qquad 0 \le x \le y \le 1, \\ \frac{1}{15120}y(x-1)(-32x+56xy^2-42xy^4+56x^2y^2-84x^3y^2 \\ +21x^4y^2+21x^2y^4+3y^6-32x^2+24x^3+24x^4-18x^5+3x^6), \\ \qquad 0 \le y < x \le 1. \end{cases} \tag{4.3}$$

Calculations show that

$$\int_0^1 dx\, K_2(x,x) = \frac{1}{90}, \int_0^1 dx\, K_3(x,x) = \frac{1}{945}, \int_0^1 dx\, K_4(x,x) = \frac{1}{9450}, \tag{4.4}$$

and the values $\zeta(4)$, $\zeta(6)$, and $\zeta(8)$ quickly follow. However, this approach of directly calculating $K_n(\cdot,\cdot)$ for each positive integer n does not seem to lend itself to a general formula.

Alternatively, for $n \in \mathbb{N}$ and $y \ge x$, $K_n(x,y)$ satisfies the boundary value problem

$$\begin{aligned} &\frac{\partial^2 h(x,y)}{\partial x^2} = -K_{n-1}(x,y), \\ &h(0,y) = h(x,1) = \frac{\partial h}{\partial x}(0,0) = 0; \end{aligned} \tag{4.5}$$

consequently, knowing $K_{n-1}(\,\cdot\,,\,\cdot\,)$ allows us to determine $K_n(\cdot,\cdot)$. This approach, as well, is not helpful in determining $K_n(\cdot,\cdot)$ for all $n \in \mathbb{N}$.

Fortunately, the following considerations permit us to compute $K_n(\,\cdot\,,\,\cdot\,)$ explicitly for any $n \in \mathbb{N}$. First, we recall the general (absolutely and uniformly convergent) Green's function formula (here $I = I_{L^2((0,1);dx)}$)

$$\big((-\Delta_D)^n - zI\big)^{-1}(x, y) = K_n(z; x, y) = 2\sum_{k\in\mathbb{N}} \frac{\sin(k\pi x)\sin(k\pi y)}{[(k\pi)^{2n} - z]}, \qquad x, y \in [0, 1], \tag{4.6}$$

in terms of the normalized (simple) eigenfunctions $u_k(x) = 2^{1/2}\sin(k\pi x)$, $0 \le x \le 1$, $k \in \mathbb{N}$ of $-\Delta_D$. Indeed, equation (4.6) follows instantly from the spectral theorem for self-adjoint operators applied to the function $\big((-\Delta_D)^n - zI\big)^{-1}$ of $-\Delta_D$, yielding

$$\big((-\Delta_D)^n - zI\big)^{-1} = \sum_{k\in\mathbb{N}} \big[(k\pi)^{2n} - z\big]^{-1}(u_k, \cdot)_{\mathcal{H}} u_k. \tag{4.7}$$

Taking $z = 0$ in (4.6) yields

$$K_n(x, y) = 2\sum_{k\in\mathbb{N}} \frac{\sin(k\pi x)\sin(k\pi y)}{(k\pi)^{2n}}, \quad 0 \le x, y \le 1. \tag{4.8}$$

We are now in position to prove one of the principal results of this note.

Theorem 1 *For each* $n \in \mathbb{N}$,

$$K_n(x, y) = \begin{cases} \dfrac{(-1)^{n-1}2^{2n-1}}{(2n)!}\left[B_{2n}\left(\dfrac{y-x}{2}\right) - B_{2n}\left(\dfrac{x+y}{2}\right)\right], & 0 \le x \le y \le 1, \\[2mm] \dfrac{(-1)^{n-1}2^{2n-1}}{(2n)!}\left[B_{2n}\left(\dfrac{x-y}{2}\right) - B_{2n}\left(\dfrac{x+y}{2}\right)\right], & 0 \le y < x \le 1, \end{cases} \tag{4.9}$$

where $B_{2n}(x)$ is the Bernoulli polynomial of degree $2n$. As a corollary of (4.9) *one recovers Euler's formula for $\zeta(2n)$ (cf.* (1.9)),

$$\begin{aligned} \zeta(2n)\pi^{-2n} &= \sum_{k\in\mathbb{N}} \frac{1}{(k\pi)^{2n}} = \operatorname{tr}\big((-\Delta_D)^{-n}\big) = \int_0^1 dx\, K_n(x, x) \\ &= \int_0^1 dx\, \frac{(-1)^{n-1}2^{2n-1}}{(2n)!}\,[B_{2n} - B_{2n}(x)] \\ &= \frac{(-1)^{n-1}2^{2n-1}B_{2n}}{(2n)!}. \end{aligned} \tag{4.10}$$

Proof Since

$$\sin(k\pi x)\sin(k\pi y) = [\cos(k\pi(x-y)) - \cos(k\pi(x+y))]/2, \tag{4.11}$$

one infers from (2.6) and (4.8) that

$$
\begin{aligned}
K_n(x,y) &= \frac{2}{\pi^{2n}} \sum_{k\in\mathbb{N}} \frac{\sin(k\pi x)\sin(k\pi y)}{k^{2n}} \\
&= \frac{1}{\pi^{2n}} \sum_{k\in\mathbb{N}} \frac{\cos(k\pi(x-y)) - \cos(k\pi(x+y))}{k^{2n}} \\
&= \frac{1}{\pi^{2n}} \left[\sum_{k\in\mathbb{N}} \frac{\cos(k\pi(x-y))}{k^{2n}} - \sum_{k\in\mathbb{N}} \frac{\cos(k\pi(x+y))}{k^{2n}} \right] \\
&= \frac{1}{\pi^{2n}} \left[\frac{(-1)^{n-1}2^{2n-1}\pi^{2n}}{(2n)!} B_{2n}\left(\frac{x-y}{2}\right) - \frac{(-1)^{n-1}2^{2n-1}\pi^{2n}}{(2n)!} B_{2n}\left(\frac{x+y}{2}\right) \right] \\
&= \frac{(-1)^{n-1}2^{2n-1}}{(2n)!} \left[B_{2n}\left(\frac{x-y}{2}\right) - B_{2n}\left(\frac{x+y}{2}\right) \right]. \qquad (4.12)
\end{aligned}
$$

In particular,

$$
K_n(x,x) = \frac{(-1)^{n-1}2^{2n-1}}{(2n)!} \left[B_{2n} - B_{2n}(x) \right]. \qquad (4.13)
$$

To obtain (4.10) one employs (2.3), (4.13), and the identities in (3.4) and (3.6) applied to $T = (-\Delta_D)^{-n}$. □

5 Some Generalizations

In our final section we probe some (z-dependent) extensions of Theorem 1.

More precisely, for $x, y \in [0,1]$, $n \in \mathbb{N}$, we will be considering

$$
\big((-\Delta_D)^n - zI\big)^{-1}(x,y) = K_n(z;x,y) = 2\sum_{k\in\mathbb{N}} \frac{\sin(k\pi x)\sin(k\pi y)}{[(k\pi)^{2n} - z]}, \quad z \in \mathbb{C}\backslash\big\{(k\pi)^{2n}\big\}_{k\in\mathbb{N}}, \qquad (5.1)
$$

and

$$
\begin{aligned}
\big(-\Delta_D - zI\big)^{-n}(x,y) &= \widetilde{K}_n(z;x,y) = 2\sum_{k\in\mathbb{N}} \frac{\sin(k\pi x)\sin(k\pi y)}{[(k\pi)^2 - z]^n} \\
&= \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}} K_1(z;x,y), \quad z \in \mathbb{C}\backslash\big\{(k\pi)^2\big\}_{k\in\mathbb{N}},
\end{aligned}
$$

noting that,

$$
(-\Delta_D)^{-n}(x,y) = K_n(x,y) = K_n(z;x,y)\big|_{z=0} = \widetilde{K}_n(z;x,y)\big|_{z=0}, \qquad (5.2)
$$

$$
K_1(z;x,y) = \widetilde{K}_1(z;x,y), \quad z \in \mathbb{C}\backslash\big\{(k\pi)^2\big\}_{k\in\mathbb{N}}. \qquad (5.3)
$$

We start with the case of $\widetilde{K}_n(z;\,\cdot\,,\,\cdot\,)$, $n \in \mathbb{N}$, and recall the trace formula,

$$\begin{aligned}\sum_{k\in\mathbb{N}} \big[(k\pi)^2 - z\big]^{-\mu} &= \frac{\pi^{1/2}}{(2z^{1/2})^{\mu-(1/2)}\Gamma(\mu)} \int_0^\infty dx\, \big[e^{\pi x} - 1\big]^{-1} x^{\mu-(1/2)} I_{\mu-(1/2)}\big(z^{1/2}x\big), \qquad (5.4)\\ &\qquad \Re(\mu) > 1/2,\ \big|\Re\big(z^{1/2}\big)\big| < \pi,\end{aligned}$$

obtained as an elementary consequence of [27] [No. 6.6247] or [68] [Eq. (9) on p. 386] (originally due to Kapteyn [37]), where (cf. [46] [No. 10.27.6])

$$I_\nu(\zeta) = e^{\mp i(\pi/2)\nu} J_\nu(\pm i\zeta), \quad \Re(\nu) \geq 0,\ -\pi \leq \pm\mathrm{Arg}(\zeta) \leq \pi/2, \tag{5.5}$$

where $\mathrm{Arg}(\,\cdot\,)$ represents the single-valued principal value of the argument function on $\mathbb{C}\backslash(-\infty, 0]$. Here $J_\nu(\,\cdot\,)$ (resp., $I_\nu(\,\cdot\,)$) denotes the Bessel function (resp., modified Bessel function) of order ν (cf., e.g.., [1] [Chap. 9], [46] [Chap. 10]).

Employing

$$I_\nu(\zeta) = (\zeta/2)^\nu \sum_{k\in\mathbb{N}_0} \frac{(\zeta/2)^{2k}}{k!\,\Gamma(\nu+k+1)}, \quad \Re(\nu) \geq 0,\ \zeta \in \mathbb{C}, \tag{5.6}$$

and its asymptotics as $\zeta \to 0$, one confirms (cf. [1] [No. 23.2.7], [46] [No. 25.5.1]) that

$$\sum_{k\in\mathbb{N}} (k\pi)^{-2\mu} = \frac{1}{\Gamma(2\mu)} \int_0^\infty ds\, \big[e^{\pi s} - 1\big]^{-1} s^{2\mu-1} = \pi^{-2\mu}\zeta(2\mu), \quad \Re(\mu) > 1/2. \tag{5.7}$$

Next, one notes that the explicit formula for K_1,

$$K_1(z; x, y) = \frac{1}{z^{1/2}\sin\big(z^{1/2}\big)} \begin{cases} \sin\big(z^{1/2}x\big)\sin\big(z^{1/2}(1-y)\big), & 0 \leq x \leq y \leq 1,\\ \sin\big(z^{1/2}y\big)\sin\big(z^{1/2}(1-x)\big), & 0 \leq y \leq x \leq 1,\end{cases} \tag{5.8}$$

readily leads to the well-known fact (see (1.17) and the subsequent paragraph),

$$\begin{aligned}\sum_{k\in\mathbb{N}} \big[(k\pi)^2 - z\big]^{-1} &= \mathrm{tr}\,\big((-\Delta_D - zI_{L^2((0,1);dx)})^{-1}\big) = \int_0^1 dx\, K_1(z; x, x)\\ &= \frac{1}{2z}\big[1 - z^{1/2}\cot\big(z^{1/2}\big)\big], \quad z \in \mathbb{C}\big\backslash\big\{(k\pi)^2\big\}_{k\in\mathbb{N}}.\end{aligned} \tag{5.9}$$

(This also follows from [25] [Theorem 3.4, Example 3.15].) Here we used the elementary fact (see, e.g.., [27] [No. 2.5324]),

$$\int_0^1 dx\ \sin(ax)\sin(a(1-x)) = (2a)^{-1}\sin(a)[1-a\cot(a)], \quad a\in\mathbb{C}. \tag{5.10}$$

Lemma 1 *Let $n\in\mathbb{N}$ and $z\in\mathbb{C}\backslash\{(k\pi)^2\}_{n\in\mathbb{N}}$. Then*

$$\begin{aligned}\sum_{k\in\mathbb{N}}\big[(k\pi)^2-z\big]^{-n} &= \operatorname{tr}\big((-\Delta_D - zI_{L^2((0,1);dx)})^{-n}\big) = \int_0^1 dx\ \widetilde{K}_n(z;x,x)\\ &= \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\bigg\{\frac{1}{2z}\big[1-z^{1/2}\cot\big(z^{1/2}\big)\big]\bigg\}.\end{aligned} \tag{5.11}$$

In addition,

$$\widetilde{K}_n(z;x,y) = \sum_{m=n}^{\infty} n\binom{m}{n} K_m(x,y)\, z^{m-n}, \quad |z|<\pi^2, \tag{5.12}$$

and

$$\sum_{k\in\mathbb{N}}\big[(k\pi)^2-z\big]^{-n} = \sum_{m=n}^{\infty} n\binom{m}{n}\frac{(-1)^{m+1}2^{2m-1}B_{2m}}{(2m)!}\, z^{m-n}, \quad |z|<\pi^2. \tag{5.13}$$

Proof Differentiation of (5.9) $(n-1)$ times with respect to z yields (5.11).
Employing the resolvent expansion

$$\big((-\Delta_D)-zI\big)^{-1} = \sum_{m\in\mathbb{N}}(-\Delta_D)^{-m}z^{m-1}, \quad |z|<\pi^2, \tag{5.14}$$

yields

$$K_1(z;x,y) = \sum_{m\in\mathbb{N}} K_m(x,y)\, z^{m-1}, \quad |z|<\pi^2, \tag{5.15}$$

and hence,

$$K_{m+1}(x,y) = \frac{1}{m!}\frac{\partial^m K_1(z;x,y)}{\partial z^m}\bigg|_{z=0}. \tag{5.16}$$

Taking the trace of either side in (5.12) (cf. (4.10)) implies (5.13). □

Combining (5.4) (for $\mu = n\in\mathbb{N}$) and (5.11) then yields the evaluation of the following integral

$$\frac{\pi^{1/2}}{(2z^{1/2})^{n-(1/2)}[(n-1)!]}\int_0^\infty dx\,\big[e^{\pi x}-1\big]^{-1}x^{n-(1/2)}I_{n-(1/2)}\big(z^{1/2}x\big)$$
$$=\sum_{k\in\mathbb{N}}\big[(k\pi)^2-z\big]^{-n}=\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\bigg\{\frac{1}{2z}\big[1-z^{1/2}\cot\big(z^{1/2}\big)\big]\bigg\}, \quad (5.17)$$
$$n\in\mathbb{N},\ \big|\Re\big(z^{1/2}\big)\big|<\pi.$$

Although elementary, we were not able to find formula (5.17) in the standard literature on (integrals of) Bessel functions.

Next, we turn to a discussion of $K_n(z;\,\cdot\,,\,\cdot\,)$, $n\in\mathbb{N}$, and derive the z-dependent analogs of (4.9) and (4.10).

Theorem 2 *Let* $n\in\mathbb{N}$ *and* $z\in\mathbb{C}\backslash\big\{(k\pi)^{2n}\big\}_{n\in\mathbb{N}}$. *Then*

$$K_n(z;x,y)=\frac{1}{nz^{1-(1/n)}}\sum_{j=0}^{n-1}\omega_j K_1\big(\omega_j z^{1/n};x,y\big),\quad x,y\in[0,1], \quad (5.18)$$

and

$$\sum_{k\in\mathbb{N}}\big[(k\pi)^{2n}-z\big]^{-1}=\operatorname{tr}\big((-\Delta_D)^n-zI\big)^{-1}\big)=\int_0^1 dx\,K_n(z;x,x)$$
$$=\frac{1}{2nz}\bigg[n-\sum_{j=0}^{n-1}\omega_j^{1/2}z^{1/(2n)}\cot\big(\omega_j^{1/2}z^{1/(2n)}\big)\bigg], \quad (5.19)$$

where $\omega_j=e^{2\pi ij/n}$, $0\le j\le n-1$, *represent the nth roots of unity.*

In addition,

$$K_n(z;x,y)=\sum_{m\in\mathbb{N}}K_{nm}(x,y)\,z^{m-1},\quad |z|<\pi^{2n}. \quad (5.20)$$

and

$$\sum_{k\in\mathbb{N}}\big[(k\pi)^{2n}-z\big]^{-1}=\sum_{\ell\in\mathbb{N}_0}\frac{(-1)^{n(\ell+1)+1}2^{2n(\ell+1)-1}B_{2n(\ell+1)}}{[(2n(\ell+1)]!}z^\ell,\quad |z|<\pi^{2n}. \quad (5.21)$$

Proof We recall the partial fraction expansion

$$\frac{1}{\lambda^n-z}=\frac{1}{\prod_{j=0}^{n-1}[\lambda-\omega_j z^{1/n}]}=\frac{1}{nz^{1-(1/n)}}\sum_{j=0}^{n-1}\frac{\omega_j}{\lambda-\omega_j z^{1/n}},\quad z\in\mathbb{C}\backslash\big\{\lambda^n\big\}, \quad (5.22)$$

a consequence of the following well-known facts: Introducing

$$f_n(z) = \lambda^n - z = \prod_{j=0}^{n-1} \big(\lambda - \omega_j z^{1/n}\big), \quad z \in \mathbb{C} \tag{5.23}$$

(fixing the branch of $z^{1/n}$), one obtains

$$\begin{aligned} \frac{d}{dz} \ln(f_n(z)) &= -\big[\lambda^n - z\big]^{-1} = \frac{d}{dz} \sum_{j=0}^{n-1} \ln\big(\lambda - \omega_j z^{1/n}\big) \\ &= -\frac{1}{n} \frac{1}{z^{1-(1/n)}} \sum_{j=0}^{n-1} \omega_j \big[\lambda - \omega_j z^{1/n}\big]^{-1}, \quad z \in \mathbb{C}\backslash\big\{\lambda^n\big\}. \end{aligned} \tag{5.24}$$

Thus, by the functional calculus implied by the spectral theorem for self-adjoint operators,

$$\begin{aligned} &\big((-\Delta_D)^n - zI\big)^{-1} = \prod_{j=0}^{n-1} \big(-\Delta_D - \omega_j z^{1/n} I\big)^{-1} \\ &= \frac{1}{n z^{1-(1/n)}} \sum_{j=0}^{n-1} \omega_j \big(-\Delta_D - \omega_j z^{1/n} I\big)^{-1}, \quad z \in \mathbb{C}\backslash\big\{(k\pi)^{2n}\big\}_{n\in\mathbb{N}}, \end{aligned} \tag{5.25}$$

implying (5.18). The fact (5.10) then yields (5.19).

Relation (5.20) follows from the resolvent expansion (cf. (5.14))

$$\big((-\Delta_D)^n - zI\big)^{-1} = \sum_{m\in\mathbb{N}} (-\Delta_D)^{-nm} z^{m-1}, \quad |z| < \pi^{2n}. \tag{5.26}$$

Finally, to prove (5.21) we employ Euler's identity (cf. [28] [No. 6.87]),

$$\zeta \cot(\zeta) = \sum_{\ell\in\mathbb{N}_0} \frac{(-1)^\ell 2^{2\ell} B_{2\ell}}{(2\ell)!} \zeta^{2\ell}, \quad |\zeta| < \pi, \tag{5.27}$$

implying

$$\frac{1 - \zeta \cot(\zeta)}{2} = \sum_{\ell\in\mathbb{N}} \frac{(-1)^{\ell+1} 2^{2\ell-1} B_{2\ell}}{(2\ell)!} \zeta^{2\ell}, \quad |\zeta| < \pi. \tag{5.28}$$

Thus,

$$\sum_{k\in\mathbb{N}} \big[(k\pi)^{2n} - z\big]^{-1} = \sum_{j=0}^{n-1} \frac{1}{2nz}\big[1 - \omega_j^{1/2} z^{1/(2n)} \cot\big(\omega_j^{1/2} z^{1/(2n)}\big)\big] \tag{5.29}$$

$$= \sum_{\ell\in\mathbb{N}} \frac{1}{n}\Bigg(\sum_{j=0}^{n-1} \omega_j^{\ell}\Bigg) \frac{(-1)^{\ell+1} 2^{2\ell-1} B_{2\ell}}{(2\ell)!} z^{-1+(\ell/n)} \tag{5.30}$$

$$= \sum_{\ell\in\mathbb{N}} \frac{(-1)^{n\ell+1} 2^{2n\ell-1} B_{2n\ell}}{(2n\ell)!} z^{\ell-1}, \quad |z| < \pi^{2n}, \tag{5.31}$$

utilizing

$$\frac{1}{n}\sum_{j=0}^{n-1} \omega_j^{\ell} = \begin{cases} 1, & \ell \equiv 0 \,(\bmod\, n), \\ 0, & \text{otherwise.} \end{cases} \tag{5.32}$$

□

One notes that taking the trace on either side of (5.20) (cf. (4.10)) yields an alternative derivation of (5.21).

The cases $n = 1, 2$ in (5.19) are known and can be found in [50] [No. 5.1.25.4], [64] [p. 85], [65] [p. 113] for $n = 1$, and in [50] [No. 5.1.27.3] for $n = 2$; we have not found the cases $n \geq 3$ in the literature.

Taking $z \to 0$ in (5.21) yields another derivation of Euler's formula (4.10).

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