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ON EXTENDING THE INEQUALITIES OF PAYNE, PÓLYA, AND WEINBERGER USING SPHERICAL HARMONICS

MARK S. ASHBAUGH AND LOTFI HERMI

ABSTRACT. Using spherical harmonics, rearrangement techniques, the Sobolev inequality, and Chiti's reverse Hölder inequality, we obtain extensions of a classical result of Payne, Pólya, and Weinberger bounding the gap between consecutive eigenvalues of the Dirichlet Laplacian in terms of moments of the preceding ones. The extensions yield domain-dependent inequalities.

1. Introduction. In 1956, Payne, Pólya, and Weinberger [**43**], see also [**42**] where the results were first announced, proved that for a bounded domain $\Omega \subset \mathbf{R}^2$, the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Dirichlet eigenvalue problem for the Laplacian,

$$(1.1)\qquad \begin{aligned} -\Delta u &= \lambda u \quad \text{in} \quad \Omega,\\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

satisfy the gap inequality

$$(1.2)\qquad \lambda_{m+1} - \lambda_m \leq 2\,\frac{\sum_{i=1}^m \lambda_i}{m}, \quad \text{for} \quad m = 1, 2, 3, \dots .$$

Here multiplicities are included and thus $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$. Also, we take $u_1, u_2, u_3, \dots$ as a corresponding orthonormal basis of real eigenfunctions (in $L^2(\Omega)$).

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The result can easily be extended to cover bounded domains $\Omega \subset \mathbf{R}^n$, see [**48**], and to the setting of the Laplace-Beltrami operator on a compact hypersurface minimally immersed in $\mathbf{R}^{n+1}$ [**17**] as

$$\lambda_{m+1} - \lambda_m \leq \frac{4}{n} \frac{\sum_{i=1}^m \lambda_i}{m}. \tag{1.3}$$

In 1980, Hile and Protter [**31**] obtained this Payne, Pólya, and Weinberger (often abbreviated to PPW in what follows) inequality as a corollary to their bound

$$\sum_{i=1}^m \frac{\lambda_i}{\lambda_{m+1} - \lambda_i} \geq \frac{m}{4/n}. \tag{1.4}$$

In 1991, using a similar method of proof to that in the original PPW paper, Yang [**49**], see also [**2, 3, 8**], obtained

$$\sum_{i=1}^m (\lambda_{m+1} - \lambda_i)(n\lambda_{m+1} - (n+4)\lambda_i) \leq 0, \tag{1.5}$$

which can be written as

$$\sum_{i=1}^m (\lambda_{m+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^m \lambda_i (\lambda_{m+1} - \lambda_i) \tag{1.6}$$

to isolate the dimensional constant $4/n$ appearing in these inequalities.

All the results mentioned above are proved using the Rayleigh-Ritz principle for obtaining upper bounds for λ_{m+1}, namely,

$$\lambda_{m+1} \leq \frac{\int_\Omega \phi(-\Delta\phi)}{\int_\Omega \phi^2}, \tag{1.7}$$

provided $\phi \perp u_1, u_2, \ldots, u_m$ (ϕ, and every other function considered throughout this paper, is taken to be *real-valued*). The particular *trial functions* ϕ chosen to prove these inequalities are based on the Cartesian coordinates and lower eigenfunctions and assume the form

$$\phi_i = x_k u_i - \sum_{j=1}^m a_{ij} u_j, \tag{1.8}$$

where $a_{ij} = \int_\Omega x_k u_i u_j$ with x_k being a Cartesian coordinate (we suppress the k-dependence of the a_{ij}'s here). Summing (1.7) suitably over all coordinates $\{x_k\}_{k=1}^n$ and making appropriate use of the Cauchy-Schwarz inequality yields the above-mentioned results. More recently, Harrell and Stubbe [**29**], using a new trace formula they discovered, extended Yang's inequality to

$$\sum_{i=1}^m (\lambda_{m+1} - \lambda_i)^p \le \frac{2p}{n} \sum_{i=1}^m \lambda_i (\lambda_{m+1} - \lambda_i)^{p-1} \quad \text{for} \quad p \ge 2, \tag{1.9}$$

(see inequality (14) in [**29**, Theorem 9, page 1805]), and

$$\sum_{i=1}^m (\lambda_{m+1} - \lambda_i)^p \le \frac{4}{n} \sum_{i=1}^m \lambda_i (\lambda_{m+1} - \lambda_i)^{p-1} \quad \text{for} \quad 0 \le p \le 2, \tag{1.10}$$

see inequality (11) in [**29**, Theorem 5, page 1801]. Their results are reproved and extended to a larger class of operators in [**11**], using, essentially, the Rayleigh-Ritz method described earlier. It is also shown in [**11**] that (1.9) is weaker than Yang's inequality (1.6) if p is restricted to integer values $p > 2$. In the same paper, inequality (1.10) is shown to be intermediate between the Yang and Hile-Protter inequalities (in fact, it interpolates between them as well). One also notes the work of Levitin and Parnovski [**36**] where a connection between Harrell and Stubbe's approach to Yang's inequality (1.6) and sum rules of quantum mechanics is made.

For a survey of results stemming from the original work of Payne, Pólya, and Weinberger, see [**2, 3, 7**]. Based on (1.3), it is clear that

$$\frac{\lambda_2 - \lambda_1}{\lambda_1} \le \frac{4}{n}. \tag{1.11}$$

Payne, Pólya, and Weinberger conjectured in their work [**42, 43**] that the best bound for the quantity $(\lambda_2 - \lambda_1)/\lambda_1$ is that obtained for an n-dimensional ball, viz.

$$\frac{\lambda_2 - \lambda_1}{\lambda_1} \le \frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} - 1. \tag{1.12}$$

Here $j_{p,k}$ denotes the kth positive zero of the Bessel function $J_p(t)$ (we follow the notation of Abramowitz and Stegun [**1**] here). This optimal bound was proved by Ashbaugh and Benguria in 1991, see [**4, 5**]. In two dimensions, it is approximately equal to 1.539. Earlier, Brands [**15**] (1964) had obtained the bound 1.687, while deVries [**21**] (1967) had obtained 1.658, and Chiti [**20**] (1983) had obtained 1.586. In $\mathbf{R}^n$, Chiti's bound is given by

$$\frac{\lambda_2 - \lambda_1}{\lambda_1} \le \frac{n j_{n/2-1,1}^{-2}}{2} \frac{J_{n/2}^2(j_{n/2-1,1})}{\int_0^1 r^3 J_{n/2-1}^2(j_{n/2-1,1} r)\, dr}. \tag{1.13}$$

In [**6**], Ashbaugh and Benguria supplied the expression $6n/(2j_{n/2-1,1}^2 + n(n-4))$ as the explicit evaluation of the Chiti bound. They also gave the asymptotic expansion for their optimal bound

$$\frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} - 1 = \frac{4}{n} - \frac{4}{3}(1.8557571)\frac{2^{5/3}}{n^{5/3}} + \frac{12}{n^2} + O(n^{-7/3}). \tag{1.14}$$

For comparison, the asymptotics of the Chiti bound are given by

$$\frac{6n}{2j_{n/2-1,1}^2 + n(n-4)} = \frac{4}{n} - \frac{4}{3}(1.8557571)\frac{2^{5/3}}{n^{5/3}} + \frac{16}{n^2} + O(n^{-7/3}). \tag{1.15}$$

These bounds satisfy the inequality, see [**6**],

$$\frac{j_{n/2,1}^2}{j_{n/2-1,1}^2} - 1 < \frac{6n}{2j_{n/2-1,1}^2 + n(n-4)} < \frac{4}{n}, \tag{1.16}$$

(the latter half of this inequality was originally proved by Lee Lorch [**39**]).

The earliest "algebraization" of the PPW argument goes back to Harrell [**25**]. Hook [**33**] algebraized the original argument of Hile and Protter (herein sometimes abbreviated as HP) from [**31**] and extended it to various elliptic operators. Harrell and Michel [**27, 28**] produced a projections-based argument from which the HP and Hook results follow. Their method produced various HP-bounds for different manifolds strengthening earlier results of Harrell [**26**].

In [**7**], Ashbaugh and Benguria gave a proof of the Hile-Protter inequality which does not require the introduction of "free parameters" as in the earlier works of Hile-Protter and Hook. In [**29**], Harrell and Stubbe gave a new proof of Yang's inequalities based on commutator algebra and a new trace formula they proved.

More recently, one of us, see [**2, 3**], produced an argument based in part on the work of Yang [**49**] which avoids both "free parameters" and commutators. It constitutes a unified approach to the PPW, HP, and Yang inequalities. This proof was recently extended to produce a commutator-based "parameter-free" version of the inequalities of PPW, HP and Yang [**10**] and applied to strengthen known bounds for various elliptic operators proved earlier by Hook, Harrell, and Harrell and Michel. This latter material is presented in [**12**] where the authors apply their "unified method" to various physical and geometric spectral problems.

In this paper we will extend the PPW inequalities using spherical harmonics. So far, as described above, the inequalities obtained by various authors are *universal*: They are independent of the domain $\Omega \subset \mathbf{R}^n$. The extensions we present here provide new, *domain-dependent*, inequalities. Due to their different nature, there is no easy, direct, or general way to compare our new bounds to the previously known ones (which are domain-independent). These results are presented in Section 5. Extensions of the Hile-Protter and Yang results to domain-dependent inequalities are presented in [**9**]. In that paper we also analyze the strength of these domain-dependent inequalities.

2. Spherical harmonics. Spherical harmonics are the extension of Fourier series to dimensions $n \geq 3$. A natural way to think of them is as restrictions of homogeneous harmonic polynomials in the Cartesian coordinates to the unit $(n-1)$-sphere of $\mathbf{R}^n$. Hence, they are functions of the "angular" part of the coordinate system under consideration. For details about this class of functions, see the Bateman Manuscript Project [**22**], the books of Hochstadt [**32**], Müller [**40, 41**], Sobolev [**45**], or Axler, Bourdon, and Ramey [**14**], or Groemer's article [**24**].

The chief purpose of this section is to simplify the expression

$$\sum_S \{\nabla(gS) \cdot \nabla u\}^2$$

where the sum is taken over an orthonormal basis of real spherical harmonics of a fixed order ℓ, g is a radial function in $\mathbf{R}^n$, and both

g and u are C^1 functions on $\mathbf{R}^n$ or on some open domain $\Omega \subset \mathbf{R}^n$. The result is stated in Theorem 2.3. It will be used in our extension in Section 4.

Let $x_1, x_2, \ldots, x_n$ denote the Cartesian coordinates of a point $x \in \mathbf{R}^n$, and $e_1, e_2, \ldots, e_n$ be the standard basis of the Euclidean space. Also, let $r = |x|$ and ξ be the unit vector such that $x = r\,\xi$. In polar coordinates, x is given by [**22, 45**]

$$
\begin{aligned}
x_1 &= r\cos\theta_1,\\
x_2 &= r\sin\theta_1\cos\theta_2,\\
x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3,\\
&\ \vdots\\
x_{n-2} &= r\sin\theta_1\cdots\sin\theta_{n-3}\cos\theta_{n-2},\\
x_{n-1} &= r\sin\theta_1\cdots\sin\theta_{n-3}\sin\theta_{n-2}\cos\phi,\\
x_n &= r\sin\theta_1\cdots\sin\theta_{n-3}\sin\theta_{n-2}\sin\phi,
\end{aligned}
\tag{2.1}
$$

where $0 \le \theta_k \le \pi$ for $k = 1, 2, \ldots, n-2$ and $0 \le \phi < 2\pi$.

The gradient of a function f has the polar representation

$$
\begin{aligned}
\nabla f = {} & \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial\theta_1}\hat{\theta}_1 + \frac{1}{r\sin\theta_1}\frac{\partial f}{\partial\theta_2}\hat{\theta}_2 + \cdots\\
& + \frac{1}{r\sin\theta_1\cdots\sin\theta_{n-3}}\frac{\partial f}{\partial\theta_{n-2}}\hat{\theta}_{n-2} + \frac{1}{r\sin\theta_1\cdots\sin\theta_{n-2}}\frac{\partial f}{\partial\phi}\hat{\phi}\\
\equiv & \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\nabla_{\mathbf{S}^{n-1}} f,
\end{aligned}
\tag{2.2}
$$

where $\hat{r}, \hat{\theta}_1, \ldots, \hat{\theta}_{n-2}, \hat{\phi}$ are orthonormal vectors in the coordinate directions (in obvious notation).

The Laplace operator assumes the polar representation [**45**]

$$
\begin{aligned}
\Delta f = {} & \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\ r^{n-1}\ \frac{\partial f}{\partial r} + \frac{1}{r^2}\biggl(\frac{1}{\sin^{n-2}\theta_1}\frac{\partial}{\partial\theta_1}\ \sin^{n-2}\theta_1\ \frac{\partial f}{\partial\theta_1}\\
& + \frac{1}{\sin^2\theta_1}\frac{1}{\sin^{n-3}\theta_2}\frac{\partial}{\partial\theta_2}\ \sin^{n-3}\theta_2\ \frac{\partial f}{\partial\theta_2} + \cdots
\end{aligned}
\tag{2.3}
$$

$$+ \frac{1}{\sin^2\theta_1 \sin^2\theta_2 \cdots \sin^2\theta_{n-2}} \frac{\partial^2 f}{\partial \phi^2}\Bigg)$$

$$\equiv \frac{\partial^2 f}{\partial r^2} + \frac{n-1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\Delta_{\mathbf{S}^{n-1}} f.$$

We define $\Delta_{\mathbf{S}^{n-1}}$ to be the *spherical Laplace operator* or *spherical Laplacian* also referred to as the Laplace-Beltrami operator on $\mathbf{S}^{n-1}$ [**41**].

With this notation a spherical harmonic $S_\ell(\xi)$ of order ℓ in n dimensions satisfies

$$\Delta_{\mathbf{S}^{n-1}} S_\ell(\xi) + \ell(\ell+n-2)S_\ell(\xi) = 0. \tag{2.4}$$

The dimension of the space of spherical harmonics of order ℓ in n dimensions is

$$N_\ell = \binom{n+\ell-1}{n-1} - \binom{n+\ell-3}{n-1}$$

(with the second binomial coefficient interpreted as 0 if its lower argument exceeds its upper). It is not hard to see that N_ℓ grows like ℓ^{n-2} as $\ell \to \infty$.

Let $\Omega \subset \mathbf{R}^n$, and let $\{S_\ell^k\}_{k=1}^{N_\ell}$ denote an orthonormal family of real spherical harmonics of order ℓ and dimension n. Since these are functions on $\mathbf{S}^{n-1}$, whenever working on Ω, S_ℓ^k will mean $S_\ell^k(x/r)$ where $r = |x|$.

We now quote a theorem from the theory of spherical harmonics which will be used, in an essential way, to prove our main result in this section.

Theorem 2.1 (Addition theorem for spherical harmonics).

$$\sum_{k=1}^{N_\ell} S_\ell^k(\xi) S_\ell^k(\eta) = \frac{N_\ell}{\omega_n} P_\ell(\xi \cdot \eta), \tag{2.5}$$

where $P_\ell(t)$ is the Legendre polynomial of degree ℓ and dimension n, $\omega_n = |\mathbf{S}^{n-1}| = (2\pi^{n/2}/\Gamma(n/2))$ and $\xi, \eta \in \mathbf{S}^{n-1}$.

Proof. See [**40**] or [**41**]. $\square$

Remark. The Legendre polynomial of degree ℓ and dimension n, $P_\ell(t)$, satisfies the differential equation

$$(1-t^2)P_\ell''(t) - (n-1)tP_\ell'(t) + \ell(\ell+n-2)P_\ell(t) = 0.$$

For $t = 1$ we immediately obtain the identity

$$P_\ell'(1) = \frac{\ell(\ell+n-2)}{n-1} \tag{2.6}$$

since $P_\ell(1) = 1$ for all ℓ by definition.

We now prove a lemma which will be needed in the proofs of Theorems 2.3 and 2.4. Two alternate proofs of these sum rules for spherical harmonics are provided in the paper [**9**].

Lemma 2.2. *Let ξ be a point on the unit sphere $\mathbf{S}^{n-1}$, and let α be a unit tangent vector to $\mathbf{S}^{n-1}$ at ξ. Then,*

$$\sum_{k=1}^{N_\ell} \left(\frac{\partial S_\ell^k}{\partial \alpha}\right)^2 (\xi) = \frac{N_\ell}{\omega_n}\,\frac{\ell(\ell+n-2)}{n-1} \tag{2.7}$$

Proof. Let $\xi(t)$ be the great circle on $\mathbf{S}^{n-1}$ parametrized by the arclength starting at ξ in the direction of α. Hence, $\xi(0) = \xi$ and $\dot{\xi}(0) = \alpha$. Let

$$f(t_1, t_2) = \sum_{k=1}^{N_\ell} S_\ell^k(\xi(t_1))\ S_\ell^k(\xi(t_2)).$$

Then

$$\sum_{k=1}^{N_\ell} \left(\frac{\partial S_\ell^k}{\partial \alpha}\right)^2 (\xi) = \frac{\partial^2 f}{\partial t_1 \partial t_2}(0,0).$$

On the other hand, $\xi(t_1)\cdot\xi(t_2) = \cos(t_1 - t_2)$, so by Theorem 2.1,

$$f(t_1, t_2) = \frac{N_\ell}{\omega_n}\, P_\ell(\cos(t_1 - t_2)).$$

The statement of the lemma follows immediately from the last formula and (2.6). $\square$

Theorem 2.3. *Let $g, u \in C^1(\Omega)$, $g = g(|x|)$ be radial and $\{S_\ell^k\}_{k=1}^{N_\ell}$ an orthonormal family of real spherical harmonics of order ℓ on $\mathbf{R}^n$. Then*

(2.8)

$$\sum_{k=1}^{N_\ell} \left(\nabla(gS_\ell^k)\cdot\nabla u\right)^2 = \frac{N_\ell}{\omega_n}\left((g')^2\left(\frac{\partial u}{\partial r}\right)^2 + \frac{g^2}{r^2}\frac{\ell(\ell+n-2)}{n-1}\frac{1}{r^2}|\nabla_{\mathbf{S}^{n-1}}u|^2\right).$$

Remark. We opted to write the expression $1/r^2|\nabla_{\mathbf{S}^{n-1}}u|^2$ separately in order to emphasize the fact that this is the correct angular part of the square of the gradient in spherical coordinates. Indeed, $|\nabla u|^2 = (\partial u/\partial r)^2 + (1/r^2)|\nabla_{\mathbf{S}^{n-1}}u|^2$.

Proof. We first notice that

$$\begin{aligned}\sum_k \left(\nabla(gS_\ell^k)\cdot\nabla u\right)^2 &= (g')^2(u_r)^2\sum_k \left(S_\ell^k\right)^2 \\ &\quad + \frac{g'gu_r}{r^2}\sum_k \nabla_{\mathbf{S}^{n-1}}\left(S_\ell^k\right)^2\cdot\nabla_{\mathbf{S}^{n-1}}u \\ &\quad + \frac{g^2}{r^4}\sum_k \left(\nabla_{\mathbf{S}^{n-1}}S_\ell^k\cdot\nabla_{\mathbf{S}^{n-1}}u\right)^2 \\ &= \Sigma_1+\Sigma_2+\Sigma_3.\end{aligned}$$

By Theorem 2.1,

$$\Sigma_1 = \frac{N_\ell}{\omega_n}(g')^2(u_r)^2.$$

Moreover, $\Sigma_2 = 0$ since $\sum_k (S_\ell^k)^2$ is constant. To compute Σ_3, one applies Lemma 2.2 with $\alpha = \nabla_{\mathbf{S}^{n-1}}u/|\nabla_{\mathbf{S}^{n-1}}u|$. Indeed,

$$\sum_k \left(\nabla_{\mathbf{S}^{n-1}}S_\ell^k\cdot\nabla_{\mathbf{S}^{n-1}}u\right)^2 = |\nabla_{\mathbf{S}^{n-1}}u|^2\sum_k\left(\frac{\partial S_\ell^k}{\partial\alpha}\right)^2. \qquad \square$$

Remark. In two dimensions the result of Theorem 2.3 is easy to derive directly. First, we note that the spherical harmonic expansion is just a Fourier expansion. For $\ell \geq 1$ the orthonormal family of real spherical harmonics $\{S_l^k\}_{k=1}^{N_\ell}$ is replaced by $\{(\cos \ell\theta/\sqrt{\pi}), (\sin \ell\theta/\sqrt{\pi})\}$. For any function f, we have

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\hat{\theta}.$$

Hence, for $g = g(r)$

$$\nabla(g \cos \ell\theta) = \big(g' \cos \ell\theta\big)\hat{r} - \left(\ell \frac{g}{r} \sin \ell\theta\right)\hat{\theta}, \tag{2.9}$$

and

$$\nabla(g \sin \ell\theta) = \big(g' \sin \ell\theta\big)\hat{r} + \left(\ell \frac{g}{r} \cos \ell\theta\right)\hat{\theta}. \tag{2.10}$$

For $u \in C^1$, and $\ell \geq 1$, it follows that

$$\big(\nabla(g \cos \ell\theta) \cdot \nabla u\big)^2 + \big(\nabla(g \sin \ell\theta) \cdot \nabla u\big)^2 = (g')^2 \left(\frac{\partial u}{\partial r}\right)^2 + \ell^2 \frac{g^2}{r^4}\left(\frac{\partial u}{\partial \theta}\right)^2, \tag{2.11}$$

or

$$\sum_{k=1}^{N_\ell} \big(\nabla(g S_\ell^k) \cdot \mathcal{Q} u\big)^2 = \frac{N_\ell}{\omega_2}\left((g')^2 \left(\frac{\partial u}{\partial r}\right)^2 + \ell^2 \frac{g^2}{r^4}\left(\frac{\partial u}{\partial \theta}\right)^2\right), \tag{2.12}$$

as desired (since $N_\ell = 2$ for $\ell \geq 1$ and $\omega_2 = 2\pi$).

Theorem 2.4. *Let $g \in C^1(\Omega)$, $g = g(|x|)$ be radial and $\{S_\ell^k\}_{k=1}^{N_\ell}$ an orthonormal family of real spherical harmonics of order ℓ in $\mathbf{R}^n$. Then*

$$\sum_{k=1}^{N_\ell} \left|\nabla(g S_\ell^k)\right|^2 = \frac{N_\ell}{\omega_n}\left((g')^2 + \ell(\ell + n - 2)\ \frac{g^2}{r^2}\right). \tag{2.13}$$

Proof.

$$\begin{aligned}\nabla(g S_\ell^k) &= \frac{\partial (g S_\ell^k)}{\partial r}\hat{r} + \frac{1}{r}\nabla_{\mathbf{S}^{n-1}}(g S_\ell^k)\\ &= g'(r) S_\ell^k \hat{r} + \frac{g}{r}\nabla_{\mathbf{S}^{n-1}} S_\ell^k.\end{aligned}$$

Hence,

$$|\nabla(gS_\ell^k)|^2 = (g')^2(S_\ell^k)^2 + \frac{g^2}{r^2}|\nabla_{\mathbf{S}^{n-1}} S_\ell^k|^2.$$

Summing over all spherical harmonics yields

$$\sum_{k=1}^{N_\ell} \left|\nabla(gS_\ell^k)\right|^2 = (g')^2 \, \frac{N_\ell}{\omega_n} + \frac{g^2}{r^2} \sum_{k=1}^{N_\ell} |\nabla_{\mathbf{S}^{n-1}} S_\ell^k|^2. \tag{2.14}$$

To compute the sum on the righthand side, note that

$$|\nabla_{\mathbf{S}^{n-1}} S_\ell^k(\xi)|^2 = \sum_{j=1}^{n-1} \left(\frac{\partial S_\ell^k}{\partial \alpha_j}\right)^2 (\xi)$$

where $\alpha_1, \dots, \alpha_{n-1}$ is an orthonormal basis in the tangent space to the unit sphere at ξ, and finish by applying Lemma 2.2. □

3. Spherical harmonics extension. In their proof of the PPW conjecture, Ashbaugh and Benguria [**4, 5**] used trial functions of the form $\phi_i = P_i u_1$ for the second eigenvalue, where

$$P_i = \frac{g(r)x_i}{r} \quad \text{for } i = 1, 2, \dots, n.$$

Using (1.7) with $m = 1$, they write

$$\lambda_2 - \lambda_1 \le \frac{\int_\Omega |\nabla P|^2 u_1^2}{\int_\Omega P^2 u_1^2}.$$

Summing over all possible P_i, they obtained a "radial functional" in g (save for a mass factor of u_1^2) for the gap $\lambda_2 - \lambda_1$ of the form

$$\lambda_2 - \lambda_1 \le \frac{\int_\Omega B(r) u_1^2}{\int_\Omega g(r)^2 u_1^2}$$

where

$$B(r) = g'(r)^2 + \frac{n-1}{r^2} g(r)^2.$$

A *center of mass argument* guarantees the orthogonality conditions

$$\int_\Omega P_i u_1^2 = 0 \quad \text{for} \quad i = 1, 2, \dots, n$$

required in the Rayleigh-Ritz principle. A particular choice of $g(r)$ (given in terms of Bessel functions natural to the n-ball) and special properties of the radial functional under spherical rearrangement yields the best upper bound for the ratio of the first two eigenvalues of the fixed membrane problem. We note here that the function x_i/r is a *spherical harmonic* of order 1 in dimension n. We now generalize the method of proof used in previous works by choosing trial functions for λ_{m+1} of the form

$$\phi_i = g(r) S_\ell^k u_i - \sum_{j=1}^{m} a_{ij} u_j, \quad \text{for} \quad i = 1, 2, \dots, m. \tag{3.1}$$

Here $\{S_\ell^k\}_{k=1}^{N_\ell}$ denotes an orthonormal family of real spherical harmonics of order ℓ on $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ and, in (3.1), S_ℓ^k means $S_\ell^k(x/r)$ for $x \in \Omega$ where $r = |x|$. This is an orthonormal basis of real eigenfunctions of order ℓ, on $\mathbf{S}^{n-1}$, solutions of

$$\Delta_{\mathbf{S}^{n-1}} v + \ell(\ell + n - 2) v = 0 \tag{3.2}$$

for any fixed nonnegative integer ℓ.

In our trial functions ϕ_i we have suppressed the indices ℓ, k for simplicity. Components along $u_1, u_2, \dots, u_m$ are projected away to guarantee the condition $\phi_i \perp u_1, u_2, \dots, u_m$. Hence, the requirement

$$a_{ij} = \int_\Omega g S_\ell^k u_i u_j \, dx \quad \text{for} \quad 1 \le i, j \le m. \tag{3.3}$$

As above, we have suppressed the ℓ and k dependencies of a_{ij}.

Remark. When $m = 1$, the orthogonality condition is equivalent to choosing the origin of the coordinate system at a "weighted" center of mass of Ω, (see for example [**5**], or the more recent [**2**]).

Clearly, $a_{ij} = a_{ji}$ for $1 \le i, j \le m$. Also,

$$\int_\Omega \phi_i^2 = \int_\Omega g S_\ell^k u_i \phi_i = \int_\Omega g^2 (S_\ell^k)^2 u_i^2 - \sum_{j=1}^{m} a_{ij}^2, \tag{3.4}$$

and

$$-\Delta\phi_i = \lambda_i g S_\ell^k u_i - 2\nabla(gS_\ell^k)\cdot\nabla u_i - \Delta(gS_\ell^k)u_i - \sum_{j=1}^m a_{ij}\lambda_j u_j. \tag{3.5}$$

Therefore,

$$\int_\Omega \phi_i(-\Delta\phi_i) = \lambda_i\int_\Omega \phi_i^2 - 2\int_\Omega \phi_i\nabla(gS_\ell^k)\cdot\nabla u_i + \int_\Omega -\Delta(gS_\ell^k)u_i\phi_i. \tag{3.6}$$

Using the Rayleigh-Ritz inequality, we obtain

$$(\lambda_{m+1}-\lambda_i)\int_\Omega \phi_i^2 \le \int_\Omega \big[-2\nabla(gS_\ell^k)\cdot\nabla u_i - \Delta(gS_\ell^k)u_i\big]\phi_i. \tag{3.7}$$

By virtue of the increasing order of the λ_i's, we get

$$(\lambda_{m+1}-\lambda_m)\int_\Omega \phi_i^2 \le \int_\Omega \psi_i\phi_i, \tag{3.8}$$

where $\psi_i = -2\nabla(gS_\ell^k)\cdot\nabla u_i - \Delta(gS_\ell^k)u_i$. The Cauchy-Schwarz inequality yields

$$(\lambda_{m+1}-\lambda_m)\int_\Omega \psi_i\phi_i \le \int_\Omega \psi_i^2. \tag{3.9}$$

Finally, we sum on i, $1\le i\le m$, and over all possible "directions," i.e., for $1\le k\le N_\ell$, to obtain

$$\lambda_{m+1}-\lambda_m \le \frac{\sum_{k=1}^{N_\ell}\sum_{i=1}^m\int_\Omega(\psi_i^{\ell k})^2}{\sum_{k=1}^{N_\ell}\sum_{i=1}^m\int_\Omega \psi_i^{\ell k}\phi_i^{\ell k}}. \tag{3.10}$$

Here the dependence of ψ_i and ϕ_i on ℓ and k has been restored (and similarly for the a_{ij}'s in the proof below).

Lemma 3.1. *With notation as above,*

$$\sum_{k=1}^{N_\ell}\sum_{i=1}^m\int_\Omega \psi_i^{\ell k}\phi_i^{\ell k} = \frac{N_\ell}{\omega_n}\sum_{i=1}^m\int_\Omega\left(\frac12\Delta(g^2)+E(g)\right)u_i^2, \tag{3.11}$$

where $\omega_n = |\mathbf{S}^{n-1}| = 2\pi^{n/2}/(\Gamma(n/2))$ *and*

$$E(g) = -g\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+n-2)}{r^2}\right)g.$$

Proof. Let

$$A = \sum_{k=1}^{N_\ell}\sum_{i=1}^{m}\int_\Omega -2\phi_i^{\ell k}\nabla(gS_\ell^k)\cdot\nabla u_i$$

and

$$B = \sum_{k=1}^{N_\ell}\sum_{i=1}^{m}\int_\Omega -\Delta(gS_\ell^k)u_i\phi_i^{\ell k}.$$

We will prove that

$$A = \frac{N_\ell}{\omega_n}\sum_{i=1}^{m}\int_\Omega \frac{1}{2}\Delta(g^2)u_i^2 + \sum_{i,j=1}^{m}\sum_{k=1}^{N_\ell} a_{ij}^{\ell k}\int_\Omega -\Delta(gS_\ell^k)u_iu_j, \tag{3.12}$$

and

$$B = \frac{N_\ell}{\omega_n}\sum_{i=1}^{m}\int_\Omega E(g)u_i^2 + \sum_{i,j=1}^{m}\sum_{k=1}^{N_\ell} a_{ij}^{\ell k}\int_\Omega \Delta(gS_\ell^k)u_iu_j. \tag{3.13}$$

The lemma then follows by summing these two identities.

Starting with the definition of A, we have

$$\begin{aligned} A &= \sum_{i,k}\int_\Omega -\frac{1}{2}\nabla((gS_\ell^k)^2)\cdot\nabla(u_i^2) + 2\sum_{i,j,k} a_{ij}^{\ell k}\int_\Omega u_j\nabla(gS_\ell^k)\cdot\nabla u_i \\ &= -\frac{1}{2}\frac{N_\ell}{\omega_n}\sum_i\int_\Omega \nabla(g^2)\cdot\nabla(u_i^2) \\ &\quad + \sum_{i,j,k}\left(a_{ij}^{\ell k} + a_{ji}^{\ell k}\right)\int_\Omega u_j\nabla(gS_\ell^k)\cdot\nabla u_i \end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&= \frac{N_\ell}{\omega_n} \sum_i \int_\Omega \frac{1}{2}\Delta(g^2) u_i^2 \\
&\quad + \sum_{i,j,k} a_{ij}^{\ell k} \left(\int_\Omega u_j \nabla(g S_\ell^k) \cdot \nabla u_i + \int_\Omega u_i \nabla(g S_\ell^k) \cdot \nabla u_j \right) \\
&= \frac{N_\ell}{\omega_n} \sum_i \int_\Omega \frac{1}{2}\Delta(g^2) u_i^2 + \sum_{i,j,k} a_{ij}^{\ell k} \int_\Omega \nabla(g S_\ell^k) \cdot \nabla(u_i u_j) \\
&= \frac{N_\ell}{\omega_n} \sum_i \int_\Omega \frac{1}{2}\Delta(g^2) u_i^2 + \sum_{i,j,k} a_{ij}^{\ell k} \int_\Omega -\Delta(g S_\ell^k) u_i u_j,
\end{aligned}
$$

where we have used the symmetry of $a_{ij}^{\ell k}$ in its lower indices and have then interchanged i and j in the second half of the last summation in passing from the second to the third line. To go from the first to the second line, we used the fact that, for $\xi \in \mathbf{S}^{n-1}$,

$$
\sum_{k=1}^{N_\ell} S_\ell^k(\xi)^2 = \frac{N_\ell}{\omega_n}, \tag{3.15}
$$

see Theorem 2.1 above). To obtain the last line of (3.14), we have used Green's identity and the Dirichlet boundary conditions satisfied by the u_i's.

The case of B is immediate. Starting with the definition, it follows that

$$
B = \sum_{i,k} \int_\Omega -\Delta(g S_\ell^k)(g S_\ell^k) u_i^2 + \sum_{i,j,k} a_{ij}^{\ell k} \int_\Omega \Delta(g S_\ell^k) u_i u_j. \tag{3.16}
$$

We have

$$
\Delta(g S_\ell^k) = \left(g'' + \frac{n-1}{r} g' - \frac{\ell(\ell+n-2)}{r^2} g \right) S_\ell^k. \tag{3.17}
$$

Therefore, using (3.15) above,

$$
\sum_k -\Delta(g S_\ell^k) g S_\ell^k = -g \left(g'' + \frac{n-1}{r} g' - \frac{\ell(\ell+n-2)}{r^2} g \right) \frac{N_\ell}{\omega_n}. \tag{3.18}
$$

With $E(g)$ as defined above, the formula for B follows. $\square$

Inequality (3.9) and Lemma 3.1 allow us to write

$$\lambda_{m+1} - \lambda_m \leq \frac{\sum_{i,k} \int_\Omega \big\{ -2\nabla(gS_\ell^k) \cdot \nabla u_i - \Delta(gS_\ell^k) u_i \big\}^2}{(N_\ell/\omega_n) \sum_i \int_\Omega ((1/2)\Delta(g^2) + E(g)) u_i^2}. \tag{3.19}$$

We now restrict our study to the case when $g(r) = r^\ell$. This choice of $g(r)$ is dictated by later calculations which simplify the form of (3.19) to workable formulas. It is expected that the best we can do using this choice of g is to obtain results similar to those of Chiti [**20**]. The freedom in Ashbaugh and Benguria [**5**] in the choice of $g(r)$ (which allows them to obtain best constants) is lost. Nevertheless, results in this direction incorporate a whole range of methods not yet exploited in the context of gap bounds and offer "generalizations" of [**5**] in certain directions. The restriction on $g(r)$ makes $E(g) = 0$, essentially because $\Delta(r^\ell S_\ell^k) = 0$ since $r^\ell S_\ell^k$ is a homogeneous harmonic polynomial (note that $-g\Delta(gS_\ell^k) = E(g)S_\ell^k$). The following theorem is now proved.

Theorem 3.2. *The gap between consecutive eigenvalues of the Dirichlet Laplacian satisfies*

$$\lambda_{m+1} - \lambda_m \leq \frac{4\omega_n}{N_\ell} \frac{\sum_{i,k} \int_\Omega (\nabla(gS_\ell^k) \cdot \nabla u_i)^2}{\sum_i \int_\Omega (1/2)\Delta(g^2) u_i^2}, \tag{3.20}$$

where $\omega_n = |\mathbf{S}^{n-1}| = (2\pi^{n/2}/\Gamma(n/2))$ *and* $g = r^\ell$.

We now need to simplify the expression in (3.20). This is immediately provided by Theorem 2.3.

Theorem 3.3.

$$\lambda_{m+1} - \lambda_m \leq \frac{4\ell}{2\ell + n - 2} \frac{\sum_{i=1}^m \int_\Omega r^{2\ell-2} |\nabla u_i|^2}{\sum_{i=1}^m \int_\Omega r^{2\ell-2} u_i^2}. \tag{3.21}$$

Remark. If $\ell = 1$ we recover the PPW inequality (1.3). In this case we use $\int_\Omega |\nabla u_i|^2 = \lambda_i$ to simplify the numerator.

Proof. Using Theorems 3.2 and 2.3, we obtain

$$
\begin{aligned}
(3.22)\quad \lambda_{m+1} - \lambda_m
&\le \frac{4\sum_{i=1}^m \int_\Omega ((g')^2(\partial u_i/\partial r)^2}{\sum_{i=1}^m \int_\Omega (1/2)\Delta(g^2)u_i^2} \\
&\quad + \frac{(g^2/r^2)(\ell(\ell+n-2)/n-1)(1/r^2)|\nabla_{\mathbf{S}^{n-1}} u_i|^2)}{\sum_{i=1}^m \int_\Omega (1/2)\Delta(g^2)u_i^2}
\end{aligned}
$$

Now, $g(r) = r^\ell$ and $\Delta(r^m) = m(m+n-2)r^{m-2}$ yield $(1/2)\Delta(g^2) = \ell(2\ell+n-2)r^{2\ell-2}$, and thus

$$
\begin{aligned}
(3.23)\quad \lambda_{m+1} - \lambda_m
&\le \frac{4\sum_{i=1}^m \int_\Omega (\ell^2 r^{2\ell-2}(\partial u_i/\partial r)^2}{\sum_{i=1}^m \int_\Omega \ell(2\ell+n-2)r^{2\ell-2}u_i^2} \\
&\quad + \frac{(\ell(\ell+n-2)/(n-1))r^{2\ell-2}(1/r^2)|\nabla_{\mathbf{S}^{n-1}} u_i|^2)}{\sum_{i=1}^m \int_\Omega \ell(2\ell+n-2)r^{2\ell-2}u_i^2}
\end{aligned}
$$

Since $\ell \ge 1$ and $n \ge 2$, we see that $(\ell(\ell+n-2)/(n-1)) \le \ell^2$ and (3.21) follows. $\square$

Lemma 3.4. *Let h be a C^2 function, and let u be an eigenfunction of the Dirichlet Laplacian with corresponding eigenvalue λ on $\Omega \subset \mathbf{R}^n$. Then*

$$
\int_\Omega h(r)|\nabla u|^2 = \lambda \int_\Omega h(r)u^2 + \int_\Omega \frac{1}{2}(\Delta h)u^2. \tag{3.24}
$$

Proof. Start with Green's identity

$$
\int_\Omega \nabla u \cdot \nabla v = \int_{\partial\Omega} v\frac{\partial u}{\partial n}\, dS - \int_\Omega v\,\Delta u, \tag{3.25}
$$

where $\partial/\partial n$ indicates differentiation in the direction of the exterior normal to $\partial\Omega$. Substituting $v = hu$ and rearranging, we have

$$\begin{aligned}
\int_\Omega h(r)|\nabla u|^2 &= \int_{\partial\Omega} hu\frac{\partial u}{\partial n}\,dS - \int_\Omega hu\Delta u - \int_\Omega u\nabla u\cdot\nabla h \\
&= -\int_\Omega hu\Delta u - \frac{1}{2}\int_\Omega \nabla(u^2)\cdot\nabla h \\
&= \lambda\int_\Omega hu^2 + \frac{1}{2}\int_\Omega u^2\Delta h - \frac{1}{2}\int_{\partial\Omega} u^2\frac{\partial h}{\partial n}\,dS \\
&= \lambda\int_\Omega hu^2 + \frac{1}{2}\int_\Omega u^2\Delta h.
\end{aligned} \tag{3.26}$$

In the above we have integrated by parts and have also used $-\Delta u = \lambda u$. The Dirichlet boundary condition $u = 0$ on $\partial\Omega$ allowed us to drop the boundary terms. □

Theorem 3.5. *If $\ell \geq 2$, then*

$$\lambda_{m+1} - \lambda_m \leq \frac{4\ell}{2\ell+n-2}\left\{\frac{\sum_{i=1}^m \lambda_i \int_\Omega r^{2\ell-2}u_i^2 + (\ell-1)(2\ell+n-4)\int_\Omega r^{2\ell-4}u_i^2}{\sum_{i=1}^m \int_\Omega r^{2\ell-2}u_i^2}\right\}. \tag{3.27}$$

Proof. We apply the previous lemma to the function $h(r) = r^{2\ell-2}$ appearing in the numerator of Theorem 3.3. □

4. Rearrangement of functions. Let u be a measurable function defined on $\Omega \subset \mathbf{R}^n$, and let μ be its *distribution function* defined by $\mu(t) = |\{x \in \Omega : |u(x)| > t\}|$. The *decreasing rearrangement* of u is the function u^* defined by $u^*(s) = \inf\{t \geq 0 : \mu(t) < s\}$. The function $u^\star$ defined by $u^\star(x) = u^*(C_n|x|^n)$, where $C_n = \pi^{n/2}/\Gamma(n/2+1)$, is called the *spherically-symmetric decreasing rearrangement* of u. The *spherically-symmetric increasing rearrangement* of u, denoted $u_\star$, is defined similarly. While u^* is defined on $[0, |\Omega|]$, $u^\star$ is defined on the ball $\Omega^\star$ centered at the origin and of the same volume as Ω. The functions $|u|$, $u^\star$ and u^* are equimeasurable. Also, if $u \in L^p(\Omega)$, then

$$\int_\Omega |u|^p\,dx = \int_0^{|\Omega|} u^{*p}\,ds = \int_{\Omega^\star} u^{\star p}\,dx. \tag{4.1}$$

Lemma 4.1. *Let u be a measurable function defined in Ω, and let α be a fixed positive number. Then*

$$\frac{\int_\Omega u^2}{\int_\Omega |x|^\alpha u^2} \le \frac{\int_{\Omega^\star} u^{\star 2}}{\int_{\Omega^\star} |x|^\alpha u^{\star 2}}. \tag{4.2}$$

Proof. Because of equimeasurability, (4.2) is equivalent to

$$\int_\Omega |x|^\alpha u^2 \ge \int_{\Omega^\star} |x|^\alpha u^{\star 2}. \tag{4.3}$$

This inequality follows from the following general facts about rearrangement [**5, 20**]:

• If f and g are nonnegative functions, then

$$\int_{\Omega^\star} f^\star g^\star \, dx \ge \int_\Omega fg \, dx \ge \int_{\Omega^\star} f_\star g^\star \, dx. \tag{4.4}$$

• If $f(x) = f(|x|)$ is nonnegative and increasing then $f_\star(r) \ge f(r)$ for $0 \le r \le r^\star = \text{radius}\,(\Omega^\star)$.

Hence,

$$\int_\Omega |x|^\alpha u^2 \ge \int_{\Omega^\star} |x|_\star^\alpha u^{\star 2} \ge \int_{\Omega^\star} |x|^\alpha u^{\star 2}. \qquad \square \tag{4.5}$$

Lemma 4.2. *Suppose u is an eigenfunction of the Dirichlet Laplacian on Ω with eigenvalue λ. Then u^* is an absolutely continuous function on $[0, |\Omega|]$ and satisfies the inequality*

$$-\frac{du^*}{ds} \le \lambda n^{-2} C_n^{-2/n} s^{-2+2/n} \int_0^s u^*(t)\, dt \quad \text{a.e. on} \quad [0, |\Omega|], \tag{4.6}$$

where $C_n = (\pi^{n/2}/\Gamma(n/2+1))$.

Proof. See [**5, 46**]. $\square$

For any fixed positive λ, consider the ball $B_\lambda = \{x \in \mathbf{R}^n : |x| \leq j_{n/2-1,1}\lambda^{-1/2}\}$ where $j_{n/2-1,1}$ is the first positive zero of the Bessel function $J_{n/2-1}(t)$.

The problem $\Delta z + \mu z = 0$ in B_λ with vanishing Dirichlet boundary conditions on ∂B_λ has its first eigenvalue equal to λ. The corresponding eigenfunction is given by $z(x) = k|x|^{1-n/2} J_{n/2-1}(\lambda^{1/2}|x|)$, where k is a positive normalizing constant. This function is spherically decreasing on B_λ. To prove this, we set $t = \lambda^{1/2}|x|$ and $p = n/2 - 1$. With this notation, $z(x) = \tilde{z}(t) = \tilde{k} t^{-p} J_p(t)$, for $0 \leq t \leq j_{p,1}$, with $\tilde{k} > 0$, and $\tilde{z}'(t) = -\tilde{k} t^{-p} J_{p+1}(t)$. We have the product representation

$$J_{p+1}(t) = \frac{t^{p+1}}{2^{p+1}\Gamma(p+2)} \prod_{k=1}^{\infty} \left(1 - \frac{t^2}{j_{p+1,k}^2}\right), \tag{4.7}$$

valid for $p > -2$ and all t. Since $j_{p,1} < j_{p+1,1}$, it is clear that $J_{p+1}(t) > 0$ for $0 < t < j_{p,1}$ and hence that $\tilde{z}'(t) < 0$ there, as desired. Moreover, z satisfies

$$-\frac{dz}{ds} = \lambda n^{-2} C_n^{-2/n} s^{-2+2/n} \int_0^s z(s')\, ds' \tag{4.8}$$

when viewed as a function of $s = C_n |x|^n$.

Lemma 4.3 (Chiti's comparison theorem). *Suppose u is an eigenfunction of the Dirichlet Laplacian on Ω with eigenvalue λ, and let the function z be normalized so that $\int_\Omega u^2\, dx = \int_{B_\lambda} z^2\, dx$. Then, viewing u^* and z as functions of $s = C_n r^n$ for $s \in [0, |B_\lambda|]$, there exists $s_1 \in (0, |B_\lambda|)$ such that*

$$\begin{aligned} z(s) &\geq u^*(s) \quad \text{for } s \in [0, s_1], \\ z(s) &\leq u^*(s) \quad \text{for } s \in [s_1, |B_\lambda|]. \end{aligned} \tag{4.9}$$

Moreover, $|B_\lambda| \leq |\Omega|$.

Proof. See [**5, 18, 19, 20**]. □

Remarks. In [**5, 20**], the result of this lemma was used with u as the first eigenfunction of the Dirichlet Laplacian on Ω. The fact that it applies to any eigenfunction was established earlier by Chiti [**18, 19**]. The second statement of the lemma is a consequence of the Faber-Krahn inequality [**23, 34, 35**], see also [**16**].

Lemma 4.4. *Suppose $\alpha > 0$, and let u and z be defined as above. Then*

$$\frac{\int_{\Omega^\star} u^{\star 2}}{\int_{\Omega^\star} |x|^\alpha u^{\star 2}} \leq \frac{\int_{B_\lambda} z^2}{\int_{B_\lambda} |x|^\alpha z^2}. \tag{4.10}$$

Proof. Start with

$$\int_{\Omega^\star} |x|^\alpha u^{\star 2} = \frac{1}{C_n^{\alpha/n}} \int_0^{|\Omega|} s^{\alpha/n} u^{*2}\, ds \tag{4.11}$$

and

$$\int_{B_\lambda} |x|^\alpha z^2 = \frac{1}{C_n^{\alpha/n}} \int_0^{|B_\lambda|} s^{\alpha/n} z^2\, ds. \tag{4.12}$$

Lemma 4.3 yields

(4.13)

$$\begin{aligned}
C_n^{\alpha/n}\left(\int_{B_\lambda} |x|^\alpha z^2 - \int_{\Omega^\star} |x|^\alpha u^{\star 2}\right) \\
&= \int_0^{|B_\lambda|} s^{\alpha/n} z^2\, ds - \int_0^{|\Omega|} s^{\alpha/n} u^{*2}\, ds \\
&= \int_0^{s_1} s^{\alpha/n}\left(z^2 - u^{*2}\right) ds + \int_{s_1}^{|B_\lambda|} s^{\alpha/n}\left(z^2 - u^{*2}\right) ds \\
&\quad - \int_{|B_\lambda|}^{|\Omega|} s^{\alpha/n} u^{*2}\, ds \\
&\leq s_1^{\alpha/n} \int_0^{s_1} \left(z^2 - u^{*2}\right) ds + s_1^{\alpha/n} \int_{s_1}^{|B_\lambda|} \left(z^2 - u^{*2}\right) ds \\
&\quad - s_1^{\alpha/n} \int_{|B_\lambda|}^{|\Omega|} u^{*2}\, ds
\end{aligned}$$

$$= s_1^{\alpha/n}\left(\int_0^{|B_\lambda|} z^2\,ds - \int_0^{|\Omega|} u^{*2}\,ds\right)$$
$$= 0.$$

Thus, $\int_{\Omega^\star} |x|^\alpha u^{\star 2} \geq \int_{B_\lambda} |x|^\alpha z^2$, and the proof is complete. □

Theorem 4.5. *If u is an eigenfunction of the Dirichlet Laplacian with corresponding eigenvalue λ and α is a positive constant, then*

$$\frac{\int_\Omega u^2}{\int_\Omega |x|^\alpha u^2} \leq \frac{j_{n/2-1,1}^{-\alpha}}{2} \frac{J_{n/2}^2(j_{n/2-1,1})}{\int_0^1 r^{\alpha+1} J_{n/2-1}^2(j_{n/2-1,1} r)\,dr}\,\lambda^{\alpha/2}. \tag{4.14}$$

Proof. We combine Lemmas 4.1 and 4.4. Observing the fact that $dx = r^{n-1}\,d\sigma\,dr$, where $d\sigma$ represents the canonical measure on $\mathbf{S}^{n-1}$, we calculate

$$\begin{aligned} \frac{\int_{B_\lambda} z^2\,dx}{\int_{B_\lambda} |x|^\alpha z^2\,dx} &= \frac{\int_0^{j_{n/2-1,1}/\sqrt{\lambda}} \int_{\mathbf{S}^{n-1}} r J_{n/2-1}^2(\sqrt{\lambda}\,r)\,d\sigma\,dr}{\int_0^{j_{n/2-1,1}/\sqrt{\lambda}} \int_{\mathbf{S}^{n-1}} r^{\alpha+1} J_{n/2-1}^2(\sqrt{\lambda}\,r)\,d\sigma\,dr} \\ &= \frac{\int_0^{j_{n/2-1,1}/\sqrt{\lambda}} r J_{n/2-1}^2(\sqrt{\lambda}\,r)\,dr}{\int_0^{j_{n/2-1,1}/\sqrt{\lambda}} r^{\alpha+1} J_{n/2-1}^2(\sqrt{\lambda}\,r)\,dr}. \end{aligned} \tag{4.15}$$

Substituting $t = \sqrt{\lambda}\,r$ yields

$$\begin{aligned} \frac{\int_{B_\lambda} z^2\,dx}{\int_{B_\lambda} |x|^\alpha z^2\,dx} &= \frac{\int_0^{j_{n/2-1,1}} t J_{n/2-1}^2(t)\,dt}{\int_0^{j_{n/2-1,1}} t^{\alpha+1} J_{n/2-1}^2(t)\,dt}\,\lambda^{\alpha/2} \\ &= \frac{\int_0^1 r J_{n/2-1}^2(j_{n/2-1,1} r)\,dr}{\int_0^1 r^{\alpha+1} J_{n/2-1}^2(j_{n/2-1,1} r)\,dr}\left(\frac{\sqrt{\lambda}}{j_{n/2-1,1}}\right)^\alpha. \end{aligned} \tag{4.16}$$

The proof is completed by observing, as in [**5**], that

$$\int_0^{j_{n/2-1,1}} t J_{n/2-1}^2(t)\,dt = \frac{1}{2} j_{n/2-1,1}^2 J'_{n/2-1}(j_{n/2-1,1})^2, \tag{4.17}$$

and $tJ_p'(t) = -tJ_{p+1}(t) + pJ_p(t)$. Hence,

(4.18)
$$\int_0^{j_{n/2-1,1}} tJ_{n/2-1}^2(t)\,dt = \frac{1}{2}\,j_{n/2-1,1}^2 J_{n/2}^2(j_{n/2-1,1}),$$

and

(4.19)
$$\int_0^1 rJ_{n/2-1}^2(j_{n/2-1,1}r)\,dr = \frac{1}{2}J_{n/2}^2(j_{n/2-1,1}),$$

which, along with (4.16), gives the bound (4.14). □

Corollary 4.6. *If u is normalized we obtain*

$$\int_\Omega |x|^\alpha u^2 \geq C_{n,\alpha}\lambda^{-\alpha/2}, \tag{4.20}$$

where

$$C_{n,\alpha} = 2j_{n/2-1,1}^\alpha\ \frac{\int_0^1 r^{\alpha+1}J_{n/2-1}^2(j_{n/2-1,1}r)\,dr}{J_{n/2}^2(j_{n/2-1,1})}.$$

Remark. Chiti's approach [**20**] (which was followed in [**4, 5**] to get the best bound for the ratio of the first two eigenvalues) avoids the Cauchy-Schwarz inequality used in passing from (3.8) to (3.9). The trial functions for λ_2 used in [**20**] were $x_i u_1$ for $i = 1, 2, \ldots, n$. The origin was chosen so that it lay at the center of mass via the requirement $\int_\Omega x_i u_1^2 = 0$. This choice avoids the coefficients a_{ij} used to project away lower eigenfunctions in (1.8) or (3.1) and assures orthogonality. The Rayleigh-Ritz principle yields

$$\lambda_2 \leq \frac{\int_\Omega |\nabla(x_i u_1)|^2\,dx}{\int_\Omega x_i^2 u_1^2\,dx}. \tag{4.21}$$

Summing suitably gives

$$\lambda_2 \leq \frac{\int_\Omega \sum_{i=1}^n |\nabla(x_i u_1)|^2\,dx}{\int_\Omega |x|^2 u_1^2\,dx} = \frac{\lambda_1 \int_\Omega |x|^2 u_1^2\,dx + n\int_\Omega u_1^2\,dx}{\int_\Omega |x|^2 u_1^2\,dx}, \tag{4.22}$$

or

$$\lambda_2 - \lambda_1 \le n \frac{\int_\Omega u_1^2\,dx}{\int_\Omega |x|^2 u_1^2\,dx}. \tag{4.23}$$

Using Corollary 4.6 with $\alpha = 2$ and $\lambda = \lambda_1 = \lambda_1(\Omega)$ yields Chiti's bound (1.13).

5. New inequalities for the eigenvalues of the Dirichlet-Laplacian. In this section we find explicit upper estimates for $\int_\Omega r^{2\ell-2}u^2$ and $\int_\Omega r^{2\ell-4}u^2$ in terms of the eigenvalue λ and geometric properties of the region Ω. These bounds will enable us to arrive at general inequalities relating various moments of the first m eigenvalues to the geometry of Ω. We note that these two integrals are compatible in the form in which they appear in (3.27) since $\lambda \propto (\text{length})^{-2}$. In general, we will deal with $\int_\Omega r^\alpha u^2$ where α is a fixed positive number and u is an eigenfunction of the Dirichlet Laplacian associated with the eigenvalue λ. In prior work, see [**5, 7, 20**], such integrals have been dealt with using rearrangement. However, this method is not useful in handling the integrals in the numerator of the righthand side of (3.27) since $g(r) = r^\alpha$ is an increasing function and straightforward rearrangement would provide lower rather than upper bounds for these integrals. Rearrangement is, of course, useful in handling the integral in the denominator of the righthand side of (3.27), and in this we follow the prior work alluded to above.

In subsections 5.1, 5.2 and 5.3, we present three alternatives for overcoming this difficulty. They provide explicit upper bounds for $\lambda_{m+1} - \lambda_m$ in terms of various moments of the preceding eigenvalues and various higher-order moments of the region Ω.

5.1 The Sobolev alternative (for $n \ge 3$). Applying Hölder's inequality we get

$$\int_\Omega r^\alpha u^2 \le \left(\int_\Omega r^{\alpha p}\right)^{1/p} \left(\int_\Omega u^{2q}\right)^{1/q}, \tag{5.1}$$

with $1/p + 1/q = 1$ and $p, q \ge 1$.

Theorem 5.1 (Sobolev's inequality for gradients). *For $n \ge 3$, let f be a sufficiently smooth function which vanishes at infinity. Then*

$f \in L^q(\mathbf{R}^n)$ with $q = 2n/(n-2)$ and the inequality

$$\left(\int |f|^q\right)^{2/q} \leq \frac{1}{S_n}\int |\nabla f|^2, \tag{5.2}$$

holds with

$$S_n = \frac{n(n-2)}{4}|\mathbf{S}^n|^{2/n} = \frac{n(n-2)}{4}2^{2/n}\pi^{1+1/n}\Gamma\left(\frac{n+1}{2}\right)^{-2/n}. \tag{5.3}$$

Equality holds if and only if f is a multiple of $\left(\mu^2 + |x-a|^2\right)^{-(n-2)/2}$ with $\mu > 0$ and $a \in \mathbf{R}^n$ arbitrary.

Proof. See [**38**]. □

Remarks. This is the Sobolev inequality in its *sharp form.* This theorem appears in the works of Aubin [**13**], Lieb [**37**] and Talenti [**46**] (see also [**47**]). The sharp bound and case of equality are due to Talenti [**46**] (see also [**38**]). Note that in the expression for S_n the factor $|\mathbf{S}^n|^{2/n}$ (rather than the seemingly more natural $|\mathbf{S}^{n-1}|^{2/n}$) is *not* a misprint.

We let $2q = 2n/(n-2)$ in (5.1) and use the theorem for the eigenfunction u with eigenvalue λ. This makes $p = n/2$ and

$$\int_\Omega r^\alpha u^2 \leq \frac{\lambda}{S_n}\left(\int_\Omega r^{\alpha n/2}\right)^{2/n} \tag{5.4}$$

since u is assumed to be a normalized Dirichlet eigenfunction of $-\Delta$ on Ω and therefore $\int_\Omega |\nabla u|^2 = \lambda$.

Let $I_\alpha = \int_\Omega r^\alpha\, dx$. If $\alpha = 2$, I_α is just the usual *second moment* of Ω. For $\alpha \geq 2$, it constitutes a higher-order moment of the region Ω. It is easy to calculate in the case of a sphere. Combining Theorem 3.5, Corollary 4.6 and estimate (5.4), we obtain the following theorem.

Theorem 5.2. *For $n \geq 3$ and $\ell \geq 2$, the eigenvalues of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbf{R}^n$ satisfy the inequality*

$$(5.5)\quad \begin{aligned}(\lambda_{m+1}-\lambda_m)\left(\sum_{i=1}^{m}\frac{1}{\lambda_i^{\ell-1}}\right) &\leq \frac{4\ell}{(2\ell+n-2)S_n C_{n,2\ell-2}} \\ &\times\left(I_{(\ell-1)n}^{2/n}\sum_{i=1}^{m}\lambda_i^2+(\ell-1)(2\ell+n-4)I_{(\ell-2)n}^{2/n}\sum_{i=1}^{m}\lambda_i\right),\end{aligned}$$

with

$$C_{n,2\ell-2}=\frac{2j_{n/2-1,1}^{2\ell-2}\int_0^1 r^{2\ell-1}J_{n/2-1}^2(j_{n/2-1,1}r)\,dr}{J_{n/2}^2(j_{n/2-1,1})}$$

and with S_n as given in Theorem 5.1.

5.2 Chiti alternative I. Starting with $\int_\Omega r^\alpha u^2$, we first apply the Cauchy-Schwarz inequality and then couple it with a reverse Hölder inequality result due to Chiti [**19**]. This method leads to an alternative to Theorem 5.2 with generally higher powers of the eigenvalues and factors of lower (and potentially more accessible) geometric moments of the region Ω.

Theorem 5.3. (Chiti [**19**]). *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Let λ be an eigenvalue of the Dirichlet Laplacian and u a corresponding eigenfunction. If p and q are real positive numbers such that $q \geq p > 0$, then*

$$(5.6)\qquad \left(\int |u|^q\right)^{1/q} \leq \lambda^{n/2(1/p-1/q)}K(p,q,n)\left(\int |u|^p\right)^{1/p},$$

where

$$(5.7)\quad \begin{aligned}K(p,q,n) &= (nC_n)^{(1/q-1/p)} \\ &\times \frac{\left(\int_0^{j_{n/2-1,1}} r^{n-1+q(1-n/2)}J_{n/2-1}^q(r)\,dr\right)^{1/q}}{\left(\int_0^{j_{n/2-1,1}} r^{n-1+p(1-n/2)}J_{n/2-1}^p(r)\,dr\right)^{1/p}}\end{aligned}$$

$$= (nC_n)^{(1/q-1/p)} j_{n/2-1,1}^{n(1/q-1/p)} \times \frac{(\int_0^1 r^{n-1+q(1-n/2)} J_{n/2-1}^q (j_{n/2-1,1} r) dr)^{1/q}}{(\int_0^1 r^{n-1+p(1-n/2)} J_{n/2-1}^p (j_{n/2-1,1} r) dr)^{1/p}}.$$

Equality holds if and only if $p = q$ or Ω is a sphere and λ is the first eigenvalue associated with the problem.

Proof. See [**19**]. □

By the Cauchy-Schwarz inequality, we have

$$\int_\Omega r^\alpha u^2 \le \left(\int_\Omega r^{2\alpha}\right)^{1/2} \left(\int_\Omega u^4\right)^{1/2}. \tag{5.8}$$

We apply Chiti's reverse Hölder inequality with $p = 2$ and $q = 4$ to obtain

$$\int_\Omega r^\alpha u^2 \le K(2,4,n)^2 \left(\int_\Omega r^{2\alpha}\right)^{1/2} \lambda^{n/4}. \tag{5.9}$$

Coupled with Theorem 3.5 and Corollary 4.6 we obtain the following theorem.

Theorem 5.4. *For all positive integers n, $\ell \ge 2$, the eigenvalues of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbf{R}^n$ satisfy the inequality*

$$\begin{aligned}(\lambda_{m+1} - \lambda_m)\left(\sum_{i=1}^m \frac{1}{\lambda_i^{\ell-1}}\right) &\le \frac{4\ell K(2,4,n)^2}{(2\ell+n-2)C_{n,2\ell-2}} \\ &\times \left(I_{4\ell-4}^{1/2} \sum_{i=1}^m \lambda_i^{n/4+1} + (\ell-1)(2\ell+n-4) I_{4\ell-8}^{1/2} \sum_{i=1}^m \lambda_i^{n/4}\right),\end{aligned} \tag{5.10}$$

where $C_{n,2\ell-2}$ and $K(2,4,n)$ are as given above.

5.3 Chiti alternative II. An alternative to the use of the Cauchy-Schwarz inequality in the previous section is to first apply Hölder's inequality and then follow it by Chiti's reverse Hölder inequality and send q to ∞. In this subsection we apply this idea to develop further eigenvalue inequalities. As a corollary, we derive inequalities relating eigenvalue gaps to moments of the preceding eigenvalues and to the volume and second moment of the domain Ω; see Corollary 5.6.

Start with (5.1). We then apply Chiti's reverse Hölder inequality to obtain

$$\left(\int_\Omega u^{2q}\right)^{1/q} \le K^2(2,2q,n)\lambda^{n(1/2-(1/2q))}. \tag{5.11}$$

The Bessel function $J_\nu(t)$ satisfies

$$t^{-\nu}J_\nu(t) = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{2^{2k+\nu}k!\Gamma(1+\nu+k)}, \tag{5.12}$$

with the series on the right being convergent for all t. Since $t^{-\nu}J_\nu(t)$ is a decreasing function, for $0 \le t \le j_{\nu,1}$ (see the argument following Lemma 4.2 above), we obtain, by comparing with its value at $t=0$,

$$t^{-\nu}J_\nu(t) \le \frac{1}{2^\nu\Gamma(1+\nu)}, \tag{5.13}$$

from which we derive

$$\begin{aligned} K(2,2q,n) &\le \frac{\big(nC_n\big)^{(1/2q)-1/2}\Big((1/n)j^n_{n/2-1,1}\Big)^{1/(2q)}}{2^{n/2-1}\Gamma(n/2)\big(\int_0^{j_{n/2-1,1}} rJ^2_{n/2-1}(r)\,dr\big)^{1/2}} \\ &= \frac{\big(C_n j^n_{n/2-1,1}\big)^{1/(2q)}}{2^{n/2-1}\Gamma(n/2)\big(nC_n\int_0^{j_{n/2-1,1}} rJ^2_{n/2-1}(r)\,dr\big)^{1/2}}. \end{aligned} \tag{5.14}$$

If we now take the limit as $q\to\infty$, we find

$$K(2,\infty,n) \le \frac{1}{2^{n/2-1}\Gamma(n/2)\big(nC_n\int_0^{j_{n/2-1,1}} rJ^2_{n/2-1}(r)\,dr\big)^{1/2}}. \tag{5.15}$$

In fact, the righthand side of (5.15) is the limit of $K(2,2q,n)$ as $q\to\infty$ (which is what denote by $K(2,\infty,n)$), so that (5.15) is actually an equality (just use the fact that the righthand side of (5.13) gives the ∞-norm of $t^{-\nu}J_\nu(t)$ for $0<t<j_{\nu,1}$). In this case, we must take $p=1$ in (5.1), and we obtain

$$\int_\Omega r^\alpha u^2 \le K_n \lambda^{n/2} \int_\Omega r^\alpha, \tag{5.16}$$

where, using (4.18), we have

$$K_n = K(2,\infty,n)^2 = \frac{2}{nC_n 2^{n-2}\Gamma(n/2)^2 j_{n/2-1,1}^2 J_{n/2}^2(j_{n/2-1,1})}. \tag{5.17}$$

Combining (5.16) with Theorem 3.5 and Corollary 4.6, we obtain the following theorem.

Theorem 5.5. *For all positive integers n, $\ell \ge 2$, the eigenvalues of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbf{R}^n$ satisfy the inequality*

$$\begin{aligned}(\lambda_{m+1}-\lambda_m)\left(\sum_{i=1}^m \frac{1}{\lambda_i^{\ell-1}}\right) &\le \frac{4\ell K_n}{(2\ell+n-2)C_{n,2\ell-2}}\\ &\times\left(I_{2\ell-2}\sum_{i=1}^m \lambda_i^{n/2+1} + (\ell-1)(2\ell+n-4)I_{2\ell-4}\sum_{i=1}^m \lambda_i^{n/2}\right),\end{aligned} \tag{5.18}$$

with $C_{n,2\ell-2}$ and K_n as defined above.

If $\ell=2$, I_2 is the second moment of Ω and $I_0=|\Omega|$. This implies the following corollary.

Corollary 5.6. *The eigenvalues of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbf{R}^n$ satisfy*

$$\begin{aligned}(\lambda_{m+1}-\lambda_m)\left(\sum_{i=1}^m \frac{1}{\lambda_i}\right) \le \frac{8K_n}{(n+2)C_{n,2}}\left(I_2\sum_{i=1}^m \lambda_i^{n/2+1} + n|\Omega|\sum_{i=1}^m \lambda_i^{n/2}\right).\end{aligned} \tag{5.19}$$

Remark. Theorem 5.5 could have been obtained using yet another result due to Chiti [**18**]. In this work, it will serve as a means to double check the constants we obtained in these calculations.

Theorem 5.7 (Chiti [**18**]). *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Let λ be an eigenvalue of the Dirichlet Laplacian on Ω and u a corresponding eigenfunction. Then*

(5.20)
$$\operatorname{ess\,sup}|u| \le \left(\frac{\lambda}{\pi}\right)^{n/4} \frac{2^{1-n/2}}{\Gamma(n/2)^{1/2} j_{n/2-1,1} J_{n/2}(j_{n/2-1,1})} \left(\int_\Omega u^2\right)^{1/2}.$$

Proof. Start with equation (5.6). Set $p = 2$ and send q to infinity. The result follows immediately. □

Remark. A detailed proof, from first principles, is given in [**18**].

This theorem implies the inequality

$$\int_\Omega r^\alpha u^2 \le \frac{\lambda^{n/2}}{\pi^{n/2}} \frac{2^{2-n}}{\Gamma(n/2) j^2_{n/2-1,1} J^2_{n/2}(j_{n/2-1,1})} I_\alpha. \tag{5.21}$$

Noting that $\pi^{n/2} = (nC_n\Gamma(n/2))/2$ and substituting in (5.21) yields (5.16) with the same factor K_n.

5.4 Two-dimensional Sobolev alternative. One advantage of following the works of Chiti is that we are able to obtain inequalities relating moments of the domain Ω to the gap and certain sums over eigenvalues which hold for all dimensions $n \ge 2$. This is not the case for the Sobolev alternative, which applies only for $n \ge 3$. There is, however, a different form of the *Sobolev inequality for gradients* in the case $n = 2$.

Theorem 5.8 (Sobolev's inequality for gradients in $\mathbf{R}^2$). *Let $f \in H^1(\mathbf{R}^2)$ and $2 \le q < \infty$, then*

$$\|f\|_q^2 \le \frac{1}{S_{2,q}} \big(\|\nabla f\|_2^2 + \|f\|_2^2\big), \tag{5.22}$$

where

$$S_{2,q}^{-1} = \left(\frac{q-2}{8\pi}\right)^{1-2/q} \frac{q^{2-4/q}}{(q-1)^{2-2/q}}, \tag{5.23}$$

for $q > 2$ *and* $S_{2,2} = 1$, *the limiting value as* $q \to 2$.

Proof. See [**38**] where the constant $S_{2,q}$ should be adjusted as noted here. □

Replacing the term $\left(\int_\Omega u^{2q}\right)^{1/q}$ in (5.1) by setting $f = u$ in this theorem is not convenient since it gives an upper bound equal to $S_{2,q}^{-1}(\lambda + 1)$ and there is no obvious way of comparing the energy term λ with 1, the normalization constant for $\|u\|_2$. In order to circumvent this difficulty, we use the following modification of this theorem (which is certainly also well known).

Theorem 5.9. *Let* $f \in H^1(\mathbf{R}^2)$ *and* $2 \le q < \infty$, *then*

$$\|f\|_q \le L_q \;\|f\|_2^{2/q}\; \|\nabla f\|_2^{1-2/q}, \tag{5.24}$$

with

$$L_q = \frac{q^{3/2-2/q}(q-1)^{-1+1/q}}{2^{1/q}(8\pi)^{1/2-1/q}} \tag{5.25}$$

and $L_2 = 1$.

Proof. Assume $q > 2$. We start with the statement of Theorem 5.8, and apply it to the function $v = f(x/k)$ where $k > 0$ is a constant and where, for simplicity, we set

$$C = S_{2,q}^{-1}.$$

Therefore,

$$\|v\|_q^2 \le C\big(\|\nabla v\|_2^2 + \|v\|_2^2\big). \tag{5.26}$$

A change of variable takes us back to f, since it is defined on all of $\mathbf{R}^2$, now with

$$\|f\|_q^2 \leq C\big(k^{-4/q}\|\nabla f\|_2^2 + k^{2-4/q}\ \|f\|_2^2\big). \tag{5.27}$$

The righthand side is a function of k which takes its minimum at the value

$$k_1 = \frac{\sqrt{2}}{\sqrt{q-2}}\ \frac{\|\nabla f\|_2}{\|f\|_2}.$$

This gives

$$\|f\|_q \leq \sqrt{C}2^{-1/q}q^{1/2}(q-2)^{-1/2+1/q}\|f\|_2^{2/q}\|\nabla f\|_2^{1-2/q}. \tag{5.28}$$

The desired inequality follows from substitution of the value of C in this last statement. □

Remark. Talenti describes the ideas behind the method used in this proof and many other Sobolev-type inequalities in [**47**].

Now, we apply Theorem 5.9 with $2q$ replacing q and $q \geq 1$ (so that $2q \geq 2$) to u, an eigenfunction of the Dirichlet Laplacian with corresponding eigenvalue λ, to obtain

$$\left(\int_\Omega u^{2q}\right)^{1/q} = \|u\|_{2q}^2 \leq L_{2q}^2\lambda^{1-(1/q)}. \tag{5.29}$$

Using Hölder's inequality (5.1) with $p = q' = q/(q-1)$ we arrive at

$$\int_\Omega r^\alpha u^2 \leq L_{2q}^2 I_{\alpha p}^{1/p}\ \lambda^{1-(1/q)}. \tag{5.30}$$

Using Theorem 3.5 and Corollary 4.6 (with $n = 2$ in both), we thus obtain a Sobolev version of Theorem 5.2 in two dimensions.

Theorem 5.10. *For $\ell \geq 2$, $q \geq 1$, and $p = q' = q/(q-1)$, the eigenvalues of the Dirichlet Laplacian on a bounded domain $\Omega \subset \mathbf{R}^2$ satisfy the inequality*

(5.31)

$$\big(\lambda_{m+1} - \lambda_m\big)\left(\sum_{i=1}^m \frac{1}{\lambda_i^{\ell-1}}\right)$$

$$\le \frac{2L_{2q}^2}{C_{2,2\ell-2}}\left(I_{(2\ell-2)p}^{1/p}\sum_{i=1}^{m}\lambda_i^{2-(1/q)}+2(\ell-1)^2 I_{(2\ell-4)p}^{1/p}\sum_{i=1}^{m}\lambda_i^{1-(1/q)}\right),$$

with $C_{2,2\ell-2}$ *and* L_{2q} *as defined above.*

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Department of Mathematics, University of Missouri, Columbia, MO 65211-0001
Email address: mark@math.missouri.edu

Department of Mathematics, University of Arizona, 617 Santa Rita, Tucson, AZ 85721
Email address: hermi@math.arizona.edu