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MAX-TO-MEAN RATIO ESTIMATES FOR THE FUNDAMENTAL EIGENFUNCTION OF THE DIRICHLET LAPLACIAN

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Abstract. We review upper and lower bound isoperimetric properties of the fundamental eigenfunction of the Dirichlet Laplacian and announce new reverse Hölder type inequalities for norms of this function in the case of a wedge-like membrane.

Contents

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1. Introduction

The following article serves two purposes. In the first instance, we review various isoperimetric mean-to-peak ratio inequalities and related inequalities estimating various norms of the fundamental mode of vibration of the fixed membrane problem. We also announce recent improvements in the case of a wedge-like membrane. Salient features that go into the proof of these inequalities, and some known extensions, will be highlighted.

To fix the notation, we let $\Omega \subset \mathbb{R}^d$, and consider the first eigenfunction of the Dirichlet Laplacian and its associated eigenvalue, which we will denote respectively by u and $\lambda > 0$. This problem is described by

$$\begin{aligned} \Delta u + \lambda u &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

It is well known that u can be taken to be positive. We are interested in sharp isoperimetric inequalities relating the various norms $\|u\|_\infty = \operatorname{ess\,sup} u$ (which we will denote sometimes as u_{max}), $\|u\|_p, \|u\|_q$ (for $q \geq p > 0$) to the eigenvalue λ and underlying geometric features such as volume $|\Omega|$ and surface area $|\partial\Omega|$. Here $\|u\|_m = \left(\int_\Omega u^m\, dx\right)^{1/m}$.

The oldest result we are aware of is attributed in the work of Titchmarsh (1958) to Minakshisundaram (1942) (see p. 190 of [44]) who proved that $\exists\, C$ such that

$$\|u\|_\infty \leq C\lambda^{(d-1)/4}, \tag{1.2}$$

under the normalizing condition $\|u\|_2 = 1$. Minakshisundaram's proof uses an asymptotic technique reminiscent of the work of Titchmarsh which will not be discussed further.

Properly defined, mean-to-peak ratio for the fundamental eigenfunction of the Dirichlet Laplacian was first defined by Payne and Stakgold [37, 43, 55] as

$$E = \frac{\int_\Omega u\, dx}{|\Omega|\, u_{max}}$$

in the context of shape optimization for a nuclear reactor operating at criticality. Indeed, for a homogeneous, monoenergetic, critical reactor, the neutron density, u, is the first eigenfunction of the Dirichlet Laplacian described in (1.1). The physical motivation offered [37] for studying its properties is that high mean-to-peak neutron density ratio guarantees adequate average power output without exceeding the maximum temperature due to metallurgical considerations. Schaefer and Sperb [49, 50] call this quantity "neutron density ratio". Sperb [52] simply calls it "efficiency ratio". We are interested in both upper and lower isoperimetric bounds for this ratio, and more generally for ratios of the form

$$\frac{\|u\|_q}{\|u\|_p}$$

where $q \geq p > 0$. This includes the case $q = \infty$. Rather than expressing known results in terms of E, we will simply write them in terms of u and λ.

Bounds for such ratios emulate two well-known classical isoperimetric inequalities, namely the Rayleigh-Faber-Krahn inequality [7, 42] and the Makai-Pólya inequality [30, 41] (see also [24, 48]). The first states that λ is minimized for the unit ball in $\mathbb{R}^d$ (or unit disk when $d = 2$) of same volume (or area) as Ω,

$$\lambda \geq \frac{C_d^{2/d}\, j_{d/2-1,1}^2}{|\Omega|^{2/d}} \tag{1.3}$$

where $j_{d/2-1,1}$ denotes the first positive zero of the Bessel function $J_{d/2-1}(x)$ and C_d the volume of the unit ball. The second, valid when Ω is convex, takes the form

$$\lambda \leq \frac{\pi^2}{4}\, \frac{|\partial\Omega|^2}{|\Omega|^2} \tag{1.4}$$

with equality occurring, in the limit, for an infinite slab.

2. Payne-Rayner and Payne-Stakgold inequalities

For two dimensional domains, Payne and Rayner proved [34, 35] that for $\Omega \subset \mathbb{R}^2$

$$\frac{\|u\|_2^2}{\|u\|_1^2} \leq \frac{\lambda}{4\pi} \tag{2.1}$$

with equality for the disk. This is a reverse Hölder inequality. It has the interpretation of being a couched $L^2 \geq 4\pi A$ classical isoperimetric inequality for the domain Ω with a conformal metric based on the eigenfunction u (see [12] for details). The proof of this theorem relies on Schwarz symmetrization (or decreasing rearrangement), the Faber-Krahn inequality, and the classical geometric isoperimetric inequality. For background material on these topics, we suggest [4, 7, 11, 42, 57].

Attempting to generalize the Payne-Rayner inequality (2.1), using a new monotonicity principle for an auxiliary problem and rearrangement techniques, Kohler-Jobin [27, 28] proved several extensions to dimensions $d \geq 2$ from which she obtained the corollary

$$\frac{\|u\|_2^2}{\|u\|_1^2} \leq \frac{\lambda^{d/2}}{2d\, C_d\, j_{d/2-1,1}^{d-2}}. \tag{2.2}$$

Among the many results obtained by Kohler-Jobin [27, 28] we mention the explicit isoperimetric inequalities (see (2.5) below)

$$\frac{\|u\|_\infty}{\|u\|_1} \leq C_1(d)\lambda^{d/2} \tag{2.3}$$

and

$$\frac{\|u\|_\infty}{\|u\|_2} \leq C_2(d)\lambda^{d/4}. \tag{2.4}$$

Again, equality holds when Ω is the d-ball. The constants $C_1(d), C_2(d)$ are explicit expressions in terms of Bessel functions and their corresponding zeros. Using Schwarz symmetrization and an ingenious comparison result, Chiti [14, 15] (see also [3, 7]) was able to circumvent the auxiliary problem developed by Kohler-Jobin and proved at once

$$\frac{\|u\|_q}{\|u\|_p} \leq K(p,q,d)\, \lambda^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \text{ for } q \geq p > 0 \tag{2.5}$$

and

$$\frac{\|u\|_\infty}{\|u\|_p} \leq K(p,d)\, \lambda^{\frac{d}{2p}} \text{ for } p > 0. \tag{2.6}$$

Here

$$K(p,q,d) = (d\,C_d)^{\frac{1}{q}-\frac{1}{p}}\, j_{\frac{d}{2}-1,1}^{d\,(\frac{1}{q}-\frac{1}{p})}\, \frac{\left(\int_0^1 r^{d-1+q(1-\frac{d}{2})} J_{\frac{d}{2}-1}^{q}(j_{\frac{d}{2}-1,1}\, r)dr\right)^{\frac{1}{q}}}{\left(\int_0^1 r^{d-1+p(1-\frac{d}{2})} J_{\frac{d}{2}-1}^{p}(j_{\frac{d}{2}-1,1}\, r)dr\right)^{\frac{1}{p}}}.$$

Again, these results are isoperimetric, and equality holds when Ω is the unit ball in $\mathbb{R}^d$, and $K(p,d) = \lim_{q\to\infty} K(p,q,d)$.

Under specific conditions on the domain Ω one can improve some of these results or provide counterparts. For all convex domains $\Omega \subset \mathbb{R}^d$, Payne and Stakgold [37] proved

$$\frac{\|u\|_\infty}{\|u\|_1} > \frac{\pi}{2\,|\Omega|}. \tag{2.7}$$

The equality sign holds in the limit for an infinite slab. The proof of the Payne and Stakgold ineq. uses the interior parallels method and does not actually require the convexity of Ω, but rather that the average curvature of $\partial\Omega$ is non-negative. A weaker version of (2.7), under the same restrictions on Ω states

$$\frac{\|u\|_\infty}{\|u\|_1} > \frac{\sqrt{\lambda}}{|\partial\Omega|}. \tag{2.8}$$

These two statements are sharp in the sense that equality is assumed for an infinite slab. Note that (2.8) follows from (1.4) and (2.7). For a convex domain, Payne and Stakgold [43] also proved

$$\frac{\|u\|_\infty}{\|u\|_2} \geq \sqrt{\frac{2}{|\Omega|}} \tag{2.9}$$

and, excising information about the underlying domain,

$$\|u\|_2^2 \leq \frac{\pi}{4} \|u\|_\infty\, \|u\|_1. \tag{2.10}$$

The method of proof uses in an essential way gradient estimates, which follow from a strong version of the Hopf maximum principle, and rearrangement techniques. At the suggestion of van den Berg, Payne [32] conjectured that one ought to prove a bound of the form

$$\|u\|_\infty \leq F(|\Omega|, \|u\|_2) \tag{2.11}$$

where F is a function that is independent of the eigenvalue or other geometric quantities. This would provide a counterpart to (2.9). For a planar domain, Hersch [23] suggested the conjecture

$$\frac{\|u\|_2^2}{\|u\|_1^2} \geq \frac{\pi}{4}\,\frac{\sqrt{\lambda}}{|\partial\Omega|} \tag{2.12}$$

which would provide a counterpart to (2.1). Neither a disk, nor a rectangle, both of which were checked by Hersch, provide extrema for this inequality. Further information about this ratio for the fundamental mode of vibration can be inferred when the underlying domain Ω is star-shaped with respect to a point inside $\Omega \subset \mathbb{R}^d$. In this case, one can define the pure number of the domain, B, (see [23, 42] in terms of the "Stützfunktion", $h(\xi) = \langle \xi, n \rangle$, where ξ denotes a point on the boundary $\partial\Omega$ and n the outward normal at ξ, by

$$B = \int_{\partial\Omega} \frac{1}{h(\xi)}\, d\xi.$$

Based on the Rellich identity [47],

$$\int_{\partial\Omega} h(\xi) \left(\frac{\partial u}{\partial n}\right)^2 d\xi = 2\lambda \int_{\Omega} u^2 dx,$$

For a 2-dimensional star-shaped domain Ω, Pólya and Szegő [42] proved

$$\lambda \le j_{0,1}^2 \, \frac{B}{2|\Omega|}. \tag{2.13}$$

Equality holds if and only if, Ω is a disk centered at the origin. In the same vein, Crooke and Sperb [16] (see also [23]) proved

$$\frac{\|u\|_2^2}{\|u\|_1^2} \ge \frac{\lambda}{2B}. \tag{2.14}$$

Inequality (2.13) has been recently extended, for a star-shaped domain $\Omega \subset \mathbb{R}^d$ in [17], to

$$\lambda \le j_{d/2-1,1}^2 \, \frac{B}{d\,|\Omega|}, \tag{2.15}$$

with a proof that circumvents the use of the Rellich identity.

Two-sided bounds in the spirit of the above have been proved by several authors for generalizations of the above boundary value problem. Payne and Stakgold [43] considered the nonlinear problem

$$\begin{aligned} \Delta\, u + f(u) &= 0 \text{ in } \Omega \subset \mathbb{R}^d \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.16}$$

for a given continuous function $f(u)$, with $f(0) = 0$. This includes problem (1.1) when $f(u) = \lambda u$. Existence and uniqueness of positive solutions is laid out in [55] where a more general Robin boundary condition is also treated. Assuming smoothness conditions on the boundary, strong Hopf maximum principles are developed for a functional of u, $|\nabla u|$ which lead to useful pointwise bounds for $|\nabla u|$ in terms of u and u_{max}. When coupled with standard rearrangement arguments one is then led to Payne-Stakgold type inequalities. The results are valid for domains with nonnegative average curvature at every point of the boundary, which is the case of convex domains. For this problem, the Payne-Rayner inequality, proved in [16, 36], takes the form

$$\left(\int_{\Omega} f(u)\, dx\right)^2 \ge 8\pi \int_{\Omega} F(u)\, dx \tag{2.17}$$

where $F(u) = \int_0^u f(t)dt$. Payne, Sperb and Stakgold [36] use a different functional in the Hopf maximum part of the argument, but otherwise similar techniques, to improve some of the inequalities cited above. This is also the case of the papers of Shaefer and Sperb [49, 50, 51], and Philippin [40]. One finally points out Sperb's work [53] where an auxiliary problem which echoes earlier work by Bandle [5] and

Kohler-Jobin [27, 28] but "interpolates" differently between the torsional and membrane problems on Ω,

$$\begin{aligned} \Delta u+\lambda\left(u+\alpha\int_\Omega u\,dx\right) &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2.18}$$

where α is a fixed parameter in $(-\frac{1}{|\Omega|},\infty)$, has been treated and similar inequalities for the mean-to-max ratio and different norms of the eigenfunctions were treated. For example, the Payne Rayner inequality reads

$$\|u\|_2^2 \le \left(\frac{\lambda}{4\pi}(1+\alpha|\Omega|)^2-2\alpha\right)\|u\|_1^2. \tag{2.19}$$

In addition to the original papers [34, 35], Payne-Rayner type inequalities are now part of the standard body of literature on Schwarz symmetrization (symmetric decreasing rearrangement). Classical works include [14, 15, 25, 26, 31, 57]. Recent work has focused on extensions to non-linear eigenvalue problems [1, 2], on problems with a Gaussian weight [6], on a larger class of elliptic problems [13], on free membrane problems (with Neumann conditions) for a class of domains in $\mathbb{R}^d$ with controlled relative isoperimetric content [10], on eigenvalue problems for the p-Laplacian and pseudo-p-Laplacian [8, 9], on a Hessian eigenvalue problem [19], on deriving a bound for the pressure integral in a toroidal-plasma equilibrium [61], or for extracting geometric features for minimal surfaces as is the case in [56, 60].

Based on the results of Chiti [14, 15], van den Berg, [58] was able to prove an isoperimetric inequality relating $\|u\|_\infty$ to the inradius $\rho=\max\{\min_{y\in\partial\Omega}:|x-y|: x\in\Omega\}$ under the usual normalization condition $\|u\|_2=1$:

$$\|u\|_\infty \le C(d)\rho^{-d/2}. \tag{2.20}$$

Jazzing up the work to a spherical cap in $\mathbb{S}^{d-1}$ and sending the radius of this spherical cap to zero, he was led to the following conjecture

$$\|u\|_\infty \le C(d)\rho^{-\frac{1}{6}-\frac{d}{2}}\,D^{-\frac{1}{6}} \tag{2.21}$$

where D denotes the diameter of Ω. This conjecture of van den Berg was motivated by a counterexample of Kröger to any assertion that [29]

$$\|u\|_\infty \sim |\Omega|^{-1/2}. \tag{2.22}$$

Kröger's counterexample was a wedge $S_\alpha=\{(r,\theta): 0<r<(2\alpha/\pi)^{1/2}, 0<\theta<\pi/\alpha\}$, $\alpha\ge 1$ for which $|S_\alpha|=1$, but $\|u\|_\infty\to\infty$ as $\alpha\to\infty$. Results in this spirit appear in [59] but use probabilistic methods.

Pointwise bounds for the size of the eigenfunction have already been established by Payne and Stakgold [37] who showed, for Ω convex,

$$u(P)\le\sqrt{\lambda}\,\|u\|_\infty\,d(P,\partial\Omega) \tag{2.23}$$

and

$$u(P)\le\sqrt{\lambda}\,\|u\|_1\,\frac{d(P,\partial\Omega)}{|\Omega|} \tag{2.24}$$

where P designates the point at which u reaches its maximum. These results are related to work by Grieser and Jerison [22] who obtained estimates for both the location and size of the maximum value of u for a convex domain $\Omega\subset\mathbb{R}^2$. Their estimates compare the size and location of the ground state eigenfunction with the size and location of the ground state eigenfunction of an associated one dimensional Schrödinger operator. Grieser and Jerison have returned recently to this question for a class of domains that are very special perturbations of a rectangle [21]. We also note that Grieser [20] proved a universal statement in the same spirit of (1.2), using the wave equation method, for *any* eigenfunction of the Laplacian on a compact manifold with boundary, with Dirichlet or Neumann boundary conditions, also assuming the normalization $\|u\|_2 = 1$.

3. Inequalities for wedge-like membranes

Throughout this section, we consider a 2-dimensional domain $\Omega \subset \mathcal{W}_\alpha = \{(r,\theta) : 0 \le \theta \le \pi/\alpha\}$, $\alpha \ge 1$. In 1960, Payne and Weinberger [38] proved the curious inequality

$$\lambda \ge \lambda^* = \Big(\frac{4\alpha(\alpha+1)}{\pi}\int_\Omega h^2(r,\theta) r\, dr\, d\theta\Big)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2 \tag{3.1}$$

where $h = r^\alpha \sin \alpha\theta$. Here (r,θ) are polar coordinates taken at the apex of the wedge, and $j_{\alpha,1}$ the first zero of the Bessel function $J_\alpha(x)$. Equality holds if and only if Ω is the circular sector $\mathcal{W}_\alpha$. This inequality improves on the Faber-Krahn inequality for certain domains (as is the case of certain triangles) and has the interpretation of being a version of Faber-Krahn in dimension $2\alpha+2$ for solids of rotation [4, 33]. Note that α need not be an integer. The proof of this inequality relies on a geometric isoperimetric inequality for the quantity

$$v_0 = \int_\Omega h^2(r,\theta) r\, dr\, d\theta$$

which is optimized for the sector $\mathcal{W}_\alpha$, and a carefully crafted symmetrization argument. A counterpart inequality (3.1), involving a sectorial version of the pure number B, and isoperimetric in the sense that equality holds when $\Omega = \mathcal{W}_\alpha$, was proved by Sperb [54]. In more recent works, the Payne Weinberger inequality was extended to a wedge on the sphere $\mathbb{S}^2$ [45], and to wedges in $\mathbb{R}^d$ [46].

The natural question about a Payne-Rayner type inequality for wedge-like domain was first addressed by Philippin [39] in 1976 who proved

$$\Big(\frac{2\alpha}{\pi}\Big)^{\frac{1}{\alpha+1}} (2\alpha+2)^{-\frac{2\alpha+1}{\alpha+1}} \Big(\int_\Omega u\, h\, r dr d\theta\Big)^2 \le v_0^{\frac{\alpha}{\alpha+1}} \Big(\frac{\alpha+2}{\alpha+1}\frac{1}{\lambda^*} - \frac{1}{\lambda}\Big)\int_\Omega u^2\ r dr d\theta \tag{3.2}$$

Equality holds for the sector. This result is one of two possible extensions of Payne-Rayner suggested in higher dimensions appearing in the original paper [35] "neither of which is satisfactory". The second of such "unsatisfactory" inequalities has been proved by Mossino [31] (see also Kesavan [25, 26]) who proved a weighted isoperimetric version of (2.1) for domains $\Omega \subset \mathbb{R}^d$ involving quasi-norms in Lorentz space.

In [18] we showed that

$$\Big(\int_\Omega u^q\, h^{2-q} r dr d\theta\Big)^{\frac{1}{q}} \le K(p,q,2\alpha+2)\, \lambda^{(\alpha+1)(\frac{1}{p}-\frac{1}{q})} \Big(\int_\Omega u^p h^{2-p} r dr d\theta\Big)^{\frac{1}{p}} \tag{3.3}$$

with $K(p,q,2\alpha+2)$ as given in the Chiti statement (2.5). The proof of (3.3) is an adaptation of the method of Chiti [14, 15]. As in Payne and Weinberger [38], we first consider the transformation $u = h\,w$. The function w satisfies

$$\begin{cases} \Delta w + \frac{1}{h}\langle \nabla h, \nabla w\rangle + \lambda w = 0, & \text{in } \Omega\ ; \\ \frac{\partial w}{\partial n} = 0, & \text{on } \Gamma_1; \\ w = 0, & \text{on } \Gamma_2. \end{cases}$$

where $\Gamma_1 = \partial\Omega \cap \{\theta = 0, \frac{\pi}{\alpha}\}$. One then performs rearrangement for functionals of the form

$$\int_\Omega w^p h^2 dxdy$$

and develop a Chiti comparison theorem for sectors to complete the proof.

Payne and Stakgold pointed out in [37] that for wedge-like membranes one ought to be able to improve (2.8). However, this task has not been implemented, to the best of our knowledge.

References

[1] A. Alberico, A. Ferone, R. Volpicelli, *Some properties for eigenvalues and eigenfunctions of nonlinear weighted problems*. Rend. Mat. Appl. **19**, 45-63 (1999).

[2] A. Alvino, V. Ferone, G. Trombetti, *On the properties of some nonlinear eigenvalues*. SIAM J. Math. Anal. **29**, 437-451 (1998).

[3] M. S. Ashbaugh, L. Hermi, *On extending the inequalities of Payne, Pólya, and Weinberger using spherical harmonics*. Rocky Mountain J. Math. **38**, 1037-1072 (2008).

[4] C. Bandle, **Isoperimetric inequalities and applications**. Monographs and Studies in Mathematics, 7, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1980.

[5] C. Bandle, *Estimates for the Green's functions of elliptic operators*. SIAM J. Math. Anal. **9**, 1126-1136 (1978).

[6] M. F. Betta, F. Chiacchio, A. Ferone, *Isoperimetric estimates for the first eigenfunction of a class of linear elliptic problems*. Z. Angew. Math. Phys. **58**, 37-52 (2007).

[7] R. D. Benguria, H. Linde, *Isoperimetric inequalities for eigenvalues of the Laplace operator*. Fourth Summer School in Analysis and Mathematical Physics, pp. 1-40, Contemp. Math., 476, Amer. Math. Soc., Providence, RI, 2008.

[8] M. Belloni, V. Ferone, B. Kawohl, *Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators*. Special issue dedicated to Lawrence E. Payne. Z. Angew. Math. Phys. **54**, 771-783 (2003).

[9] G. Bognár, *Isoperimetric inequalities for some nonlinear eigenvalue problems*. Proc. Colloq. Qual. Theory Differ. Equ. **7**, Electron. J. Qual. Theory Differ. Equ., Szeged, 12pp. (2004).

[10] B. Brandolini, F. Chiacchio, C. Trombetti, *Sharp estimates for eigenfunctions of a Neumann problem*. Comm. Partial Differential Equations **34**, 1317-1337 (2009).

[11] A. Burchard, **A Short Course on Rearrangement**, Universitá di Napoli "Federico II", Preprint June 2009.

[12] T. Carroll, J. Ratzkin, *Isoperimetric inequalities and variations on Schwarz's lemma* Preprint, June 2010, arXiv:1006.2310.

[13] F. Chiacchio, *Estimates for the first eigenfunction of linear eigenvalue problems via Steiner symmetrization*. Publ. Mat. **53**, 47-71 (2009).

[14] G. Chiti, *An isoperimetric inequality for the eigenfunctions of linear second order elliptic operators*. Boll. Un. Mat. Ital. A (6) **1**, 145-151 (1982).

[15] G. Chiti, *A reverse Hölder inequality for the eigenfunctions of linear second order elliptic operators*. Z. Angew. Math. Phys. **33**, 143-148 (1982).

[16] P. S. Crooke, R. P. Sperb, *Isoperimetric inequalities in a class of nonlinear eigenvalue problems*. SIAM J. Math. Anal. **9**, 671-681 (1978).

[17] P. Freitas, D. Krejčiřík, *A sharp upper bound for the first Dirichlet eigenvalue and the growth of the isoperimetric constant of convex domains*. Proc. Amer. Math. Soc. **136**, 2997-3006 (2008).

[18] N. Gamara, A. Hasnaoui, L. Hermi, *A reverse Hölder inequality for the first eigenfunction of a wedge-like membrane*, Preprint 2011.

[19] N. Gavitone, *Isoperimetric estimates for eigenfunctions of Hessian operators*. Ric. Mat. **58**, 163-183 (2009).

[20] D. Grieser, *Uniform bounds for eigenfunctions of the Laplacian on manifolds with boundary.* Comm. Partial Differential Equations **27**, 1283-1299 (2002).

[21] D. Grieser, D. Jerison, *Asymptotics of eigenfunctions on plane domains.* Pacific J. Math. **240**, 109-133 (2009).

[22] D. Grieser, D. Jerison, *The size of the first eigenfunction of a convex planar domain.* J. Amer. Math. Soc. 11 (1998), no. 1, 4172.

[23] J. Hersch, *Isoperimetric monotonicity: some properties and conjectures (connections between isoperimetric inequalities).* SIAM Rev. **30**, 551-577 (1988).

[24] I. Joó, L. L. Stachó, *Generalization of an inequality of G. Pólya concerning the eigenfrequences of vibrating membranes.* Publ. Inst. Math. (Beograd) **31**, 65-72 (1982).

[25] S. Kesavan, **Symmetrization & applications**. Series in Analysis, 3. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006.

[26] S. Kesavan, F. Pacella, *Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalities.* Appl. Anal. **54**, 2737 (1994).

[27] M.-T. Kohler-Jobin, *Sur la première fonction propre d'une membrane: une extension à N dimensions de l'inégalité isopérimétrique de Payne-Rayner.* Z. Angew. Math. Phys. **28**, 1137-1140 (1977).

[28] M.-T. Kohler-Jobin, *Isoperimetric monotonicity and isoperimetric inequalities of Payne-Rayner type for the first eigenfunction of the Helmholtz problem.* Z. Angew. Math. Phys. **32**, 625-646 (1981).

[29] P. Kröger, *On the ground state eigenfunction of a convex domain in Euclidean space.* Potential Anal. **5**, 103-108 (1996).

[30] E. Makai, *On the fundamental frequencies of two and three dimensional membranes.* Magyar Tud. Akad. Mat. Kutató Int. Kőzl **8**, 109-123 (1963).

[31] J. Mossino, *A generalization of the Payne-Rayner isoperimetric inequality.* Boll. Un. Mat. Ital. A **2**, 335342 (1983).

[32] L. E. Payne, *Some comments on the past fifty years of isoperimetric inequalities.* Inequalities (Birmingham, 1987), 143-161, Lecture Notes in Pure and Appl. Math., 129, Dekker, New York, 1991.

[33] L. E. Payne, *Isoperimetric inequalities for eigenvalue and their applications.* Autovalori e autosoluzioni: Lectures given at a Summer School of the Centro Internazionale Matematico Estivo (C.I.M.E.) held in Chieti, Italy, August 1-9, 1962, G. Fichera (ed.), C.I.M.E. Summer Schools, Vol. 27, 1–58, (1962).

[34] L. Payne, M. E. Rayner, *Some isoperimetric norm bounds for solutions of the Helmholtz equation.* Z. Angew. Math. Phys. **24**, 105-110 (1973).

[35] L. Payne, M. E. Rayner, *An isoperimetric inequality for the first eigenfunction in the fixed membrane problem.* Z. Angew. Math. Phys. **23**, 13-15 (1972).

[36] L. E. Payne, R. Sperb, I. Stakgold, *On Hopf type maximum principles for convex domains.* Nonlinear Anal. **1**, 547-559 (1976/77).

[37] L. Payne, I. Stakgold, *On the mean value of the fundamental mode in the fixed membrane problem.* Collection of articles dedicated to Alexander Weinstein on the occasion of his 75th birthday. Applicable Anal. **3**, 295-306 (1973).

[38] L. E. Payne, H. F. Weinberger, *A Faber-Krahn inequality for wedge-like membranes.* J. Math. and Phys. **39**, 182-188 (1960).

[39] G. A. Philippin, *Some isoperimetric norm bounds for the first eigenfunction of wedge-like membranes.* Z. Angew. Math. Phys. **27**, 545-551 (1976).

[40] G. A. Philippin, *On the first eigenfunction of the fixed membrane: some extensions of results of Payne and Stakgold.* Z. Angew. Math. Phys. **28**, 151-159 (1977).

[41] G. Pólya, *Two more inequalities between physical and geometrical quantities.* J. Indian Math. Soc.**24**, 413-419 (1960).

[42] G. Pólya, G. Szegő, **Isoperimetric Inequalities in Mathematical Physics**. Princeton Univ. Press, 1951.

[43] I. Stakgold, L. E. Payne, *Nonlinear problems in nuclear reactor analysis*, **Nonlinear problems in the physical sciences and biology**, Lecture Notes in Mathematics, Vol. 322, 298–307 (1973).

[44] E. C. Titchmarsh, **Eigenfunction Expansions**, Vol. 2, Oxford University Press, 1958.

[45] J. Ratzkin, A. Treibergs, *A Payne-Weinberger eigenvalue estimate for wedge domains on spheres.* Proc. Amer. Math. Soc. 137 (2009), no. 7, 22992309.

[46] J. Ratzkin, *Eigenvalues of Euclidean wedge domains in higher dimensions*, Calc. Var. Partial Differential Equations, November 2010, DOI: 10.1007/s00526-010-0381-8.

[47] F. Rellich, *Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral.* Math. Z. *46*, 635-636 (1940).

[48] A. Savo, *Lower bounds for the nodal length of eigenfunctions of the Laplacian.* Ann. Global Anal. Geom. 19 (2001), no. 2, 133151.

[49] P. W. Schaefer, R. P. Sperb, *A maximum principle for a class of functionals in nonlinear Dirichlet problems.* Ordinary and partial differential equations (Proc. Fourth Conf., Univ. Dundee, Dundee, 1976), pp. 400-406. Lecture Notes in Math., Vol. 564, Springer, Berlin, 1976.

[50] P. W. Schaefer, R. P. Sperb, *Maximum principles and bounds in some inhomogeneous elliptic boundary value problems.* SIAM J. Math. Anal. *8*, 871-878 (1977); Corrigendum and addendum, SIAM J. Math. Anal. **10**, 1326 (1979).

[51] P. W. Schaefer, R. P. Sperb, *Maximum principles for some functionals associated with the solution of elliptic boundary value problems.* Arch. Rational Mech. Anal. **61**, 65-76 (1976).

[52] R. Sperb, **Maximum principles and their applications**. Mathematics in Science and Engineering, 157. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[53] R. Sperb, *On an eigenvalue problem arising in chemistry.* Z. Angew. Math. Phys. **32**, 450-463 (1981).

[54] R. Sperb, *Extension of an inequality of Pólya-Szegő to wedge-like domain.* Z. Angew. Math. Phys. **44**, 173–177 (1981).

[55] I. Stakgold, *Global estimates for nonlinear reaction and diffusion.* Ordinary and partial differential equations (Proc. Conf., Univ. Dundee, Dundee, 1974), pp. 252-266. Lecture Notes in Math., Vol. 415, Springer, Berlin, 1974.

[56] F. Takahashi, A. Uegaki, *A Payne-Rayner type inequality for the Robin problem on arbitrary minimal surfaces in $\mathbb{R}^N$.* Osaka City University Advanced Mathematical Institute (OCAMI) Preprint, April 2010.

[57] G. Talenti, *Elliptic equations and rearrangements.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **3**, 697-718 (1976).

[58] M. van den Berg, *On the L^∞ norm of the first eigenfunction of the Dirichlet Laplacian.* Potential Anal. 13 (2000), no. 4, 36136.

[59] M. van den Berg, E. Bolthausen, *Estimates for Dirichlet eigenfunctions.* J. London Math. Soc. **59**, 607619 (1999).

[60] Q. Wang, C. Xia, *Isoperimetric bounds for the first eigenvalue of the Laplacian.* Z. Angew. Math. Phys. **61**, 171-175 (2010).

[61] Z. Yoshida, Y. Giga, *Bound for the pressure integral in a toroidal-plasma equilibrium.* Phys. Rev E **48**, 2133-2135 (1993).

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