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Isoperimetric Inequalities for a Wedge-Like Membrane

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Abstract. For a wedge-like membrane, Payne and Weinberger proved in 1960 an isoperimetric inequality for the fundamental eigenvalue which in some cases improves the classical isoperimetric inequality of Faber–Krahn. In this work, we introduce "relative torsional rigidity" for this type of membrane and prove new isoperimetric inequalities in the spirit of Saint-Venant, Pólya–Szegő, Payne, Payne–Rayner, Chiti, and Talenti, which link the eigenvalue problem with the boundary value problem in a fundamental way.

1. Introduction and Main Results

In this paper, we extend some classical results [39,41–44,48] focusing on fundamental modes of vibration of wedge-like membranes. These are two-dimensional domains contained in a sector. We concentrate on isoperimetric results, rather than asymptotics.

We announce new results focusing on the interplay between the fundamental mode of vibration of a fixed wedge-like membrane and "relative torsional rigidity", which we introduce for such domains. Along the way, a Saint-Venant-type principle for "relative torsional rigidity" is proved, as well as several other isoperimetric inequalities for the spectral problem, and for the Dirichlet boundary value problem in the spirit of Talenti, which are of independent interest. Further isoperimetric inequalities of the type of Pólya–Szegő, Kohler-Jobin, and Payne for relative torsional rigidity appear in [19]. Much of these results are extended to convex cones in higher dimensions in [20] (see also [44]).

For the purposes of the problem we are treating, we let $\alpha \geq 1$ and $\mathcal{W}$ be the wedge defined in polar coordinates (r, θ) by

$$\mathcal{W} = \left\{ (r, \theta) \;\middle|\; r > 0,\ 0 < \theta < \frac{\pi}{\alpha} \right\}. \tag{1.1}$$

Whenever pertinent, the arclength will be denoted by

$$\mathrm{d}\sigma = \sqrt{\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2},$$

while the element of area is denoted by $\mathrm{d}A = r\mathrm{d}r\mathrm{d}\theta$, and we let

$$h(r, \theta) = r^{\alpha} \sin \alpha\theta. \tag{1.2}$$

Then, h is a positive harmonic function in $\mathcal{W}$ which is zero on the boundary $\partial\mathcal{W}$.

We are interested in the first eigenfunction u and the corresponding eigenvalue λ of the problem

$$\mathcal{P}_1 : \begin{cases} \Delta u + \lambda u = 0 & \text{in } D \\ u = 0 & \text{on } \partial D, \end{cases}$$

where D is a bounded domain completely contained in $\mathcal{W}$.

We are also interested in a quantity we call "relative torsional rigidity", P_α, the existence and properties of which are detailed in Sect. 5.

Our work hovers around four classical inequalities which serve both as motivation, and means of discovery, via an interpretation in Weinstein fractional space: the Faber–Krahn inequality for the fundamental eigenvalue of a fixed membrane, λ, the Saint-Venant principle for "torsional rigidity", P, first investigated in 1856 [15], and proved by Pólya in 1948 [37,38] for two-dimensional domains, and two inequalities investigated by Pólya and Szegő in 1951 [38] for $P\,\lambda^2$ and $P\,\lambda$ ($P\,\lambda^{(d+2)/2}$ and $P\,\lambda$ for d-dimensional domains). For a wedge-like membrane $D \subset \mathcal{W}$, Payne and Weinberger [39] proved in 1960 (see also [34])

$$\lambda \geq \lambda^* = \left(\frac{4\alpha(\alpha+1)}{\pi} \int_D h^2(r, \theta) r \, \mathrm{d}r \, \mathrm{d}\theta \right)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2 \tag{1.3}$$

where h is defined via (1.2) and $j_{\alpha,1}$ denotes the first positive zero of the Bessel function of the first kind $J_\alpha(x)$ in the notation of [1]. Equality in (1.3) holds if and only if D is a circular sector.

We complete the program in [39] by proving similar results for P_α, $P_\alpha \lambda^{\alpha+2}$, and $P_\alpha\,\lambda$.

In Sect. 2, we motivate our results via an interpretation of the four classical inequalities discussed in Weinstein $(2\alpha + 2)$ fractional space in the spirit of Payne [34] (see also [39]).

In Sect. 3, we prove new versions of Chiti's isoperimetric inequality for weighted L_p-norms of the fundamental eigenfunction u of a fixed membrane problem for $D \subset \mathcal{W}$, and exploit its consequences.

In Sect. 4, we focus on the Dirichlet boundary value problem for this type of domain. We extend earlier work of Maderna and Salsa [31] and prove weighted versions of Talenti [51].

In Sect. 5, we focus on "relative torsional rigidity," state its various formulations, prove the Saint-Venant principle for this quantity [37], and extend earlier work by Pólya–Szegő [38], and Payne–Rayner [35,36]. Our results complete earlier work of Philippin [41]. All new inequalities lend themselves to the $(2\alpha + 2)$ interpretation in fractional Weinstein space which we discuss next.

2. Payne Interpretation in Weinstein Fractional Space

We first recall key properties of λ and P which will be used in this section when the finite volume domain $D \subset \mathbb{R}^d$. The fundamental eigenvalue of the Dirichlet Laplacian is characterized by the Rayleigh–Ritz principle

$$\lambda = \inf_{\phi \in W_0^{1,2}(D)} \frac{\int_D |\nabla \phi|^2 \, \mathrm{d}x}{\int_D \phi^2 \, \mathrm{d}x} \tag{2.1}$$

where $W_0^{1,2}(D)$ denotes the usual Sobolev space on D. In the case of torsional rigidity the Rayleigh–Ritz principle takes the form

$$\frac{1}{P} = \inf_{\phi \in W_0^{1,2}(D)} \frac{\int_D |\nabla \phi|^2 \, \mathrm{d}x}{\left(\int_D \phi \, \mathrm{d}x\right)^2}. \tag{2.2}$$

The optimizer v of (2.2), usually called the stress (or "warping") function, [46] satisfies

$$\Delta v = -1 \quad \text{in } D, \quad v = 0 \quad \text{on } \partial D, \tag{2.3}$$

and thus P (sometimes denoted $P(D)$ for emphasis) also admits the following forms:

$$P = \int_D v \, \mathrm{d}x = \int_D |\nabla v|^2 \, \mathrm{d}x. \tag{2.4}$$

We next list the four inequalities discussed earlier when $D \subset \mathbb{R}^d$. We will need them for the purposes of this section. The Faber–Krahn inequality [22,28] states that

$$\lambda(D) \geq \lambda(D^*) = \frac{C_d^{2/d} \, j_{d/2-1,1}^2}{|D|^{2/d}} \tag{2.5}$$

where $C_d = \pi^{d/2}/\Gamma(1 + d/2)$ is the volume of the unit ball, and D^* is the symmetrization of D, i.e., the ball with the same volume as D, viz.,

$$|D| = |D^*|.$$

The Saint-Venant isoperimetric inequality takes the form

$$P(D) \leq P(D^*) = \frac{|D|^{1+2/d}}{d\,(d+2)\, C_d^{2/d}}. \tag{2.6}$$

For $d = 2$, $P \leq P^* = \frac{|D|^2}{8\pi}$. In addition to Pólya's work [37], we point out other independent proofs by Makai [32] and Luttinger [30]. For higher dimension, see [4,6,23], and for inequalities involving higher eigenvalues see [53]. From (2.4), and the Rayleigh–Ritz principle for λ, it is clear that

$$P(D) = \int_D v \,\mathrm{d}x = \int_D |\nabla v|^2 \,\mathrm{d}x = \frac{\left(\int_D v \,\mathrm{d}x\right)^2}{\int_D |\nabla v|^2 \,\mathrm{d}x} \leq |D| \frac{\int_D v^2 \,\mathrm{d}x}{\int_D |\nabla v|^2 \,\mathrm{d}x} < \frac{|D|}{\lambda(D)},$$

where we used the Cauchy–Schwarz inequality to complete the argument. Therefore,

$$P(D)\,\lambda(D) < |D|. \tag{2.7}$$

This is Payne's proof of an earlier result of Pólya–Szegő [38]. The latter also conjectured the following isoperimetric inequality, which was eventually proved by Kohler-Jobin [25–27]:

$$P(D)\,\lambda(D)^{\frac{d+2}{2}} \geq P(D^*)\,\lambda(D^*)^{\frac{d+2}{2}} = C_d \,\frac{j_{\frac{d}{2}-1,1}^{d+2}}{d\,(d+2)}$$

For $d = 2$, the original conjecture takes the form [40]

$$P(D)\,\lambda^2 \geq \pi \,\frac{j_{0,1}^4}{8} = \frac{16.7\pi}{4}.$$

The closest anyone prior to Kohler-Jobin had gotten to the Pólya–Szegő conjecture were Payne and Rayner [35] who proved

$$P(D)\,\lambda^2 \geq \frac{16\pi}{4}.$$

In d-dimensions, this Payne–Rayner inequality takes the form

$$P\,\lambda^{\frac{d+2}{2}} \geq 2d\,C_d\,j_{\frac{d}{2}-1,1}^{d-2}. \tag{2.8}$$

Key to this improvement is an isoperimetric $L_2 - L_1$ result for norms of the eigenfunction u, first proved by Payne–Rayner [35], later generalized by Chiti [13]

$$\frac{\|u\|_q}{\|u\|_p} \leq K(p,q,d)\,\lambda^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for } q \geq p > 0 \tag{2.9}$$

and [14]

$$\frac{\|u\|_\infty}{\|u\|_p} \leq K(p,d)\,\lambda^{\frac{d}{2p}} \quad \text{for } p > 0. \tag{2.10}$$

Here

$$K(p,q,d) = (d\,C_d)^{\frac{1}{q}-\frac{1}{p}}\, j_{\frac{d}{2}-1,1}^{d\left(\frac{1}{q}-\frac{1}{p}\right)} \frac{\left(\int_0^1 r^{d-1+q\left(1-\frac{d}{2}\right)} J_{\frac{d}{2}-1}^q \left(j_{\frac{d}{2}-1,1}\,r\right) \mathrm{d}r\right)^{\frac{1}{q}}}{\left(\int_0^1 r^{d-1+p\left(1-\frac{d}{2}\right)} J_{\frac{d}{2}-1}^p \left(j_{\frac{d}{2}-1,1}\,r\right) \mathrm{d}r\right)^{\frac{1}{p}}}$$

and

$$K(p,d) = \lim_{q\to\infty} K(p,q,d).$$

To see (2.8), start with (2.2) and use the eigenfunction u as a test function

$$\frac{1}{P} \leq \frac{\int_D |\nabla u|^2 \,\mathrm{d}x}{\left(\int_D u \,\mathrm{d}x\right)^2} = \frac{\int_D |\nabla u|^2 \,\mathrm{d}x}{\int_D u^2 \,\mathrm{d}x} \frac{\int_D u^2 \,\mathrm{d}x}{\left(\int_D u \,\mathrm{d}x\right)^2}.$$

Applying Chiti's inequality (2.9) for $p = 1, q = 2$ and rearranging the terms leads to (2.8).

We are now ready to focus on a wedge-like membrane in light of the Payne interpretation in $(2\alpha + 2)$ Weinstein fractional space. This space is simply $\mathbb{R}^d$ with $d = 2\alpha + 2$, when the wedge-like membrane is transformed (or unfolded; see details below or in [39,54]) and is then interpreted as a solid of rotation in this Euclidean space. The technique is amply used in [42]; see also [21,39,41]. We first note that ineq. (1.3) is a case of the Faber–Krahn ineq. (2.5) in exactly this higher dimensional setting, a fact which was proved in [34]. We illustrate the procedure discussed there for the case of "relative torsional rigidity", P_α, when $\alpha = 1, 2$. We will define it, as we discuss it more fully later in Sect. 5, by:

$$P_\alpha = \int\limits_D vh\mathrm{d}A, \tag{2.11}$$

where v is a solution of the Dirichlet boundary value problem

$$-\Delta v = h \quad \text{in } D, \qquad v = 0 \quad \text{on } \partial D. \tag{2.12}$$

(a) Case $\alpha = 1$.

In this case, D is such that $y > 0$, and (2.12) reduces to

$$\Delta v + y = 0 \quad \text{in } D, \qquad v = 0 \quad \text{on } \partial D.$$

With $v = y\,w$, the problem is then

$$\Delta w + \frac{2}{y}\frac{\partial w}{\partial y} = -1 \quad \text{in } D, \qquad w = 0 \quad \text{on } \partial D \cap \{y > 0\}.$$

Let the function $\Phi(x_1, x_2, x_3, x_4)$ be defined by

$$\Phi(x_1, x_2, x_3, x_4) = w(x, y) \quad \text{where} \quad x = x_4; y = \sqrt{x_1^2 + x_2^2 + x_3^2};$$

This function has axial symmetry with respect to the x_4-axis. It is defined on

$$D_4 = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,\middle|\, x = x_4, \quad y = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad (x, y) \in D \right\}.$$

D_4 is obtained from D via rotation around the x-axis in $\mathbb{R}^4$. The function Φ satisfies

$$\Delta_4 \Phi = -1 \quad \text{in } D_4, \qquad \Phi = 0 \quad \text{on } \partial D_4.$$

Note that $d = 2\alpha + 2 = 4$. Let $\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4$. Torsional rigidity is given by

$$P = \int\limits_{D_4} \Phi \mathrm{d}V,$$

while relative torsional rigidity takes the form

$$P_1 = \int_D v\, y\, \mathrm{d}x\mathrm{d}y = \int_D w\, y^2\, \mathrm{d}x\mathrm{d}y = \frac{1}{4\pi}\int_{D_4} \Phi \mathrm{d}V = \frac{P}{4\pi}.$$

We also note that $|D_4| = (4\pi)\int_D y^2 \mathrm{d}x\mathrm{d}y$ (see [34, 47]). Therefore, applying the previous inequalities for P, we obtain

- Pólya–Szegő:

$$P < |D_4|\lambda^{-1}$$

  So

$$P_1 < \frac{1}{4\pi}\,|D_4|\lambda^{-1} = \frac{1}{4\pi}\,(4\pi)\left(\int_D y^2 \mathrm{d}x\mathrm{d}y\right)\lambda^{-1} = A_1\,\lambda^{-1}$$

  where $A_1 = \int_D y^2 \mathrm{d}x\mathrm{d}y$.
- Payne–Rayner: $P\lambda^3 \geq 8\frac{\pi^2}{2}\, j_{1,1}^2$, and therefore, $P_1\lambda^3 \geq \pi j_{1,1}^2$.
- Saint-Venant:

$$P \leq \frac{\sqrt{2}|D_4|^{3/2}}{24\pi}$$

  which implies

$$P_1 \leq \frac{1}{3}\left(\frac{1}{8\pi}\right)^{\frac{1}{2}} A_1^{3/2}.$$

  This constitutes an isoperimetric inequality for P_1 optimized by the half-disk with the same A_1 as D. The original interpretation in the case of λ, observed by Payne [34], takes the form

$$\lambda \geq \frac{1}{2}\left(\frac{\pi}{2A_1}\right)^{1/2} j_{1,1}^2$$

  and is also optimized for the half-disk.

(b) Case $\alpha = 2$.

In this case, D is such that $x > 0, y > 0$, and (2.12) reduces to

$$\Delta v + 2xy = 0 \quad \text{in } D, \quad v = 0 \quad \text{on } \partial D,$$

With $v = 2x\,y\,w$, the problem then reduces to

$$\Delta w + \frac{2}{x}\frac{\partial w}{\partial x} + \frac{2}{y}\frac{\partial w}{\partial y} = -1 \quad \text{in } D, \quad w = 0 \quad \text{on } \partial D \cap \{x > 0, y > 0\}.$$

Let the function $\Phi(x_1, x_2, x_3, y_1, y_2, y_3)$ be defined by

$$\Phi(x_1, x_2, x_3, y_1, y_2, y_3) = w(x, y)$$

with $x = \sqrt{x_1^2 + x_2^2 + x_3^2}$; $\quad y = \sqrt{y_1^2 + y_2^2 + y_3^2}$. This function is defined on

$$D_6 = \Big\{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 \Big| x = \sqrt{x_1^2 + x_2^2 + x_3^2}, y = \sqrt{y_1^2 + y_2^2 + y_3^2}, (x, y) \in D\Big\}$$

which is a domain of revolution generated by D rotating around two 3-dimensional orthogonal subspaces in $\mathbb{R}^6$. The function Φ satisfies

$$\Delta_6 \Phi = -1 \text{ in } D_6, \quad \Phi = 0 \text{ on } \partial D_6.$$

Note that $d = 2\alpha + 2 = 6$. Let $\mathrm{d}V = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3$, and

$$P = \int_{D_6} \Phi \mathrm{d}V.$$

Then

$$P_2 = 2\int_D v\,x\,y\,\mathrm{d}x\mathrm{d}y = 4\int_D w\,x^2\,y^2 \mathrm{d}x\mathrm{d}y = \frac{4}{(4\pi)^2}\int_{D_6} \Phi \mathrm{d}V = \frac{P}{4\pi^2}.$$

Also ([34,47])

$$|D_6| = \int_{D_6} \mathrm{d}V = (4\pi)^2 \int_D x^2 y^2\,\mathrm{d}x\mathrm{d}y = 4\pi^2 A_2$$

where $A_2 = 4\int_D x^2\,y^2\,\mathrm{d}x\mathrm{d}y$. Therefore, applying the previous inequalities for P, we obtain

- Pólya–Szegő:

  $$P < |D_6|\lambda^{-1}$$

  which leads to

  $$P_2 < A_2\,\lambda^{-1}.$$

- Payne–Rayner: $P\lambda^4 \geq 12\frac{\pi^3}{6}\,j_{2,1}^4$, and the corresponding inequality for relative torsional rigidity $P_2\lambda^4 \geq \frac{\pi}{2} j_{2,1}^4$.
- Saint-Venant:

  $$P \leq \frac{6^{1/3}|D_6|^{4/3}}{48\pi}$$

  which simplifies as

  $$P_2 \leq \frac{1}{4}\left(\frac{1}{72\pi}\right)^{\frac{1}{3}} A_2^{4/3}.$$

  Again the original interpretation in the case of λ was observed by Payne

  $$\lambda \geq \frac{1}{2}\left(\frac{\pi}{12A_2}\right)^{1/3} j_{2,1}^2,$$

  and isoperimetry holds for the last two inequalities for the quarter disk with the same A_2 as D. For a general $\alpha \geq 1$, the Payne interpretation holds as well (see [34]), and the results are proved in Sect. 5, independently of this trick.

3. Chiti's Theorem for a Wedge-Like Membrane

In this section we focus on the properties of the fundamental eigenfunction $u > 0$ of problem $\mathcal{P}_1$. We extend Chiti's comparison theorem [13,14] (see also [2,3,17]), given originally for domains in $\mathbb{R}^d$, and prove a weighted isoperimetric version of Chiti's key inequality for the case of a wedge-like domain D completely contained in the sector $\mathcal{W}$. We offer isoperimetric inequalities which complete and compare favorably with earlier results of Philippin [41], in the view of earlier work by Payne and Rayner [35,36]; see also [26,27]. We also explore consequences of our new inequalities. It turns out that the Payne–Weinberger inequality for wedge-like domains is a particular instance of the Chiti inequality for these domains in the limit when $q \to \infty$ and $p \to 0+$, as discussed in [13]. We first announce the key theorems, then discuss them, and finally offer the detailed proofs.

Theorem 3.1. *Let D be a bounded domain in the wedge $\mathcal{W}$. Let p, q be real numbers such that $q \geq p > 0$; then u satisfies the inequality*

$$\left(\int_D u^q h^{2-q} \mathrm{d}A\right)^{\frac{1}{q}} \leq K(p,q,\lambda,\alpha) \left(\int_D u^p\, h^{2-p} \mathrm{d}A\right)^{\frac{1}{p}} \tag{3.1}$$

with

$$K(p,q,\lambda,\alpha) = \left(\frac{\pi}{2\alpha}\right)^{\frac{p-q}{pq}} \lambda^{(\alpha+1)\frac{q-p}{pq}} \frac{\left(\int_0^{j_{\alpha,1}} r^{(2-q)\alpha+1} J_\alpha^q(r)\mathrm{d}r\right)^{\frac{1}{q}}}{\left(\int_0^{j_{\alpha,1}} r^{(2-p)\alpha+1} J_\alpha^p(r)\mathrm{d}r\right)^{\frac{1}{p}}}.$$

The result is isoperimetric in the sense that equality holds if and only if D is a circular sector of angle $\frac{\pi}{\alpha}$.

Remark 3.2. When $q = 2,\ p = 1$, we obtain the explicit form of the constant $K(2, 1, \lambda, \alpha)$, viz.

$$\int_D u^2 \mathrm{d}A \leq \frac{\alpha}{\pi j_{\alpha,1}^{2\alpha}} \lambda^{\alpha+2} \left(\int_D u\, h \mathrm{d}A\right)^2. \tag{3.2}$$

To obtain it, simply use the following integral properties of Bessel functions [1]

$$\int_0^{j_{\alpha,1}} r J_\alpha^2(r)\mathrm{d}r = \frac{j_{\alpha,1}^2}{2} J_{\alpha+1}^2(j_{\alpha,1}), \qquad \int_0^{j_{\alpha,1}} r^{\alpha+1} J_\alpha(r)\mathrm{d}r = j_{\alpha,1}^{\alpha+1} J_{\alpha+1}(j_{\alpha,1}). \tag{3.3}$$

One also notes that the $L_\infty - L_p$ version of Theorem 3.1, which has the Weinstein $(2\alpha + 2)$ fractional space interpretation of Kohler-Jobin's work [26], can be inferred as well. To state corollaries, we introduce the substitution

$$u(r,\theta) = v(r,\theta)\, h(r,\theta) \quad \text{for } (r,\theta) \in D, \tag{3.4}$$

with $v \in C^2(D)$ and vanishing on $\partial D \cap \mathcal{W}$.

Corollary 3.3. *With the same conditions given in Theorem 3.1, we have*

$$\text{ess sup}\, v \leq \frac{\alpha\lambda^{\alpha+1}}{2^{\alpha-1}\pi\Gamma(1+\alpha)j_{\alpha,1}^{\alpha+1}J_{\alpha+1}(j_{\alpha,1})}\int\limits_D uh\mathrm{d}A, \tag{3.5}$$

and

$$(\text{ess sup}\, v)^2 \leq \frac{\alpha\lambda^{\alpha+1}}{2^{2(\alpha-1)}\pi\left(\Gamma(\alpha+1)\right)^2 j_{\alpha,1}^2 J_{\alpha+1}^2(j_{\alpha,1})}\int\limits_D u^2\mathrm{d}A. \tag{3.6}$$

Proof. Sending $q \to \infty$ in ineq. (3.1) and using the fact that

$$t^{-\nu}J_\nu(t) \leq \frac{1}{2^\nu\,\Gamma(1+\nu)}$$

for $0 \leq t \leq j_{\nu,1}$ (see the details of (5.13) in [3]), we get

$$(\text{ess sup}\, v)^p \leq \frac{2\alpha\lambda^{\alpha+1}}{\left(2^\alpha\,\Gamma(\alpha+1)\right)^p\ \pi\ \int_0^{j_{\alpha,1}} r^{(2-p)\alpha+1}J_\alpha^p(r)\mathrm{d}r}\ \int\limits_D v^p\, h^2\mathrm{d}A \tag{3.7}$$

The results of the corollary for $p = 1, 2$ follow in light of (3.3). □

Remark 3.4. As pointed out in [13], the Faber–Krahn inequality, needed to prove Chiti's isoperimetric inequality, can also be seen as a limiting case of the latter when $q \to \infty$ and $p \to 0+$. This is also the case of ineq. (3.1) and the Payne–Weinberger ineq. (1.3). To see this, start with ineq. (3.7). Letting $p \to 0+$, we get

$$\int\limits_D h^2\mathrm{d}A \geq \frac{\pi j_{\alpha,1}^{2\alpha+2}}{4\alpha(\alpha+1)\lambda^{\alpha+1}}.$$

Rearranging the statement leads to the Payne–Weinberger ineq. (1.3).

We are now ready to prove our key Theorem 3.1. We perform a series of reductions before we prove this result. One should also note that Theorem 3.1 has the correct interpretation in $(2\alpha+2)$ Weinstein fractional space as well.

Recall the function v defined by (3.4). For $0 \leq t \leq \overline{v} = \text{ess sup}\, v$, let $D_t = v^{-1}((t, \overline{v}]) = \{(r,\theta) \in D | v(r,\theta) > t\}$. Define the function

$$\xi(t) = \int\limits_{D_t} h^2\mathrm{d}A \tag{3.8}$$

The co-area formula gives

$$\xi(t) = \int\limits_{D_t} h^2\mathrm{d}A = \int\limits_t^{\overline{v}}\int\limits_{\partial D_\tau}\frac{h^2}{|\nabla v|}\mathrm{d}\sigma\ \mathrm{d}\tau \tag{3.9}$$

Since D has bounded measure, the above equation shows that the function

$$t \mapsto \int\limits_{\partial D_t}\frac{h^2}{|\nabla v|}\mathrm{d}\sigma \tag{3.10}$$

is integrable, and, therefore, ξ is absolutely continuous. Hence ξ is differentiable almost everywhere and

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\int\limits_{\partial D_t} \frac{h^2}{|\nabla v|}\mathrm{d}\sigma \;<\; 0 \tag{3.11}$$

for almost all $t \in [0, \overline{v}]$. The function ξ is then a nonincreasing function and has an inverse which we denote by $t(\xi)$. In fact according to a standard result from M. A. Zareckii [50] $t(\xi)$ is absolutely continuous. Roughly speaking, our (t, ξ) correspond to (t, ζ) in Ratzkin's work [44]. What we perform here is a form of a weighted symmetrization initiated by [39]; see also [7,8,10,11,31,52]. For basic references on the general subject of symmetrization, we refer the reader to [24,51].

Note that

$$h^2 = \left(\sqrt{|\nabla v|}h\right)\left(\frac{h}{\sqrt{|\nabla v|}}\right).$$

Therefore, applying the Cauchy–Schwartz inequality we get

$$\left(\int\limits_{\partial D_t} h^2 \mathrm{d}\sigma\right)^2 \leq \left(\int\limits_{\partial D_t} \frac{h^2}{|\nabla v|}\mathrm{d}\sigma\right)\left(\int\limits_{\partial D_t} h^2|\nabla v|\mathrm{d}\sigma\right), \tag{3.12}$$

and so

$$-\,t'(\xi) = -\frac{1}{\xi'(t)} \leq \frac{\int_{\partial D_t} h^2|\nabla v|\mathrm{d}\sigma}{\left(\int_{\partial D_t} h^2\mathrm{d}\sigma\right)^2}. \tag{3.13}$$

We now use a geometrical inequality introduced by Payne and Weinberger in the following lemma:

Lemma 3.5. (Payne–Weinberger [39]). *Let $D \subset \mathcal{W}$ be a bounded domain with piecewise smooth boundary. Then*

$$\left(2\alpha\pi^{-1}\int\limits_{\partial D} h^2\mathrm{d}\sigma\right)^{\frac{2\alpha+2}{2\alpha+1}} \geq 4\pi^{-1}\alpha(\alpha+1)\int\limits_{D} h^2\mathrm{d}A. \tag{3.14}$$

Equality is attained when D is a circular sector of angle $\frac{\pi}{\alpha}$.

Using this lemma we have

$$-\,t'(\xi) \leq 4^{-\frac{\alpha}{\alpha+1}}\left(\frac{\pi}{\alpha}\right)^{-\frac{1}{\alpha+1}}(\alpha+1)^{-\frac{2\alpha+1}{\alpha+1}}\frac{\int_{\partial D_t} h^2|\nabla v|\mathrm{d}\sigma}{\xi^{\frac{2\alpha+1}{\alpha+1}}}. \tag{3.15}$$

Now, from the divergence theorem and $\Delta h = 0$, we have

$$\begin{aligned}\int\limits_{\partial D_t} h^2|\nabla v|\mathrm{d}\sigma &= -\int\limits_{D_t} div\left(h^2\nabla v\right)\mathrm{d}A\\ &= -\int\limits_{D_t} h\ \left(h\triangle v + 2\langle\nabla v, \nabla h\rangle\right)\mathrm{d}A\\ &= \lambda\int\limits_{D_t} v\,h^2\mathrm{d}A. \end{aligned}\tag{3.16}$$

Remark 3.6. $\forall p \geq 0$, we have

$$\int\limits_{D_t} v^p\,h^2\mathrm{d}A = \int\limits_t^{\overline{v}} \tau^p \int\limits_{\partial D_\tau} \frac{h^2}{|\nabla v|}\,\mathrm{d}\sigma\ \mathrm{d}\tau = -\int\limits_t^{\overline{v}} \tau^p\xi'(\tau)\mathrm{d}\tau. \tag{3.17}$$

The change of variable $\eta = \xi(\tau)$ gives

$$\int\limits_{D_t} v^p h^2\mathrm{d}A = \int\limits_0^{\xi(t)} (t(\eta))^p\,\mathrm{d}\eta. \tag{3.18}$$

Using this remark for $p = 1$ in inequality (3.15), we find

$$-\,t'(\xi) \leq \lambda\,4^{-\frac{\alpha}{\alpha+1}}\left(\frac{\pi}{\alpha}\right)^{-\frac{1}{\alpha+1}}(\alpha+1)^{-\frac{2\alpha+1}{\alpha+1}}\,\frac{\int_0^\xi t(\eta)\mathrm{d}\eta}{\xi^{\frac{2\alpha+1}{\alpha+1}}}, \tag{3.19}$$

for almost all $\xi \in [0, \xi_0]$, with $\xi_0 = \xi(0) = \int_D h^2\mathrm{d}A$.

With λ still denoting the first eigenvalue of $\mathcal{P}_1$, we consider the sector

$$S_\lambda = \left\{(r,\theta) \mid 0 < r < \frac{j_{\alpha,1}}{\sqrt{\lambda}},\ 0 < \theta < \frac{\pi}{\alpha}\right\}.$$

The eigenvalue problem in S_λ is given by

$$\mathcal{P}_2 : \begin{cases} \Delta u + \mu u = 0 & \text{in } S_\lambda\\ u = 0 & \text{on } \partial S_\lambda. \end{cases}$$

S_λ is so defined to guarantee that its first eigenvalue is equal to λ. The corresponding eigenfunction is given explicitly by

$$z(r,\theta) = h(r,\theta)\,R(r), \tag{3.20}$$

where R denotes the radial function given by

$$R(r) = cr^{-\alpha}J_\alpha(\sqrt{\lambda}r) \tag{3.21}$$

and c is a normalizing constant. For all $0 \leq s \leq \overline{R} = \text{ess sup}\,R$, let

$$S_{\lambda,s} = \left\{(r,\theta)\ \middle|\ R(r) > s,\ 0 < \theta < \frac{\pi}{\alpha}\right\} \quad \text{and} \quad \zeta(s) = \int\limits_{S_{\lambda,s}} h^2\mathrm{d}A.$$

We can now proceed exactly as in the proof of the inequality (3.19) to prove that ζ is a decreasing function and has an inverse function, which we denote by $s(\zeta)$, satisfying the integro-differential inequality

$$-\, s'(\zeta) \leq \lambda\, 4^{-\frac{\alpha}{\alpha+1}} \left(\frac{\pi}{\alpha}\right)^{-\frac{1}{\alpha+1}} (\alpha+1)^{-\frac{2\alpha+1}{\alpha+1}} \frac{\int_0^\zeta s(\eta)\mathrm{d}\eta}{\zeta^{\frac{2\alpha+1}{\alpha+1}}}, \tag{3.22}$$

for almost all $\zeta \in [0, \zeta_0]$, with $\zeta_0 = \zeta(0) = \int_{S_\lambda} h^2 \mathrm{d}A$.

Let $S_0 = \{(r,\theta) \mid 0 < r < r_0,\ 0 < \theta < \frac{\pi}{\alpha}\}$ such that

$$\int\limits_{S_0} h^2 \mathrm{d}A = \int\limits_{D} h^2 \mathrm{d}A = \xi_0. \tag{3.23}$$

An explicit computation gives that $\xi_0 = \frac{\pi}{4\alpha(\alpha+1)} r_0^{2\alpha+2}$. Now, we introduce the function $u^\star$ defined on S_0 by

$$u^\star(r,\theta) = v^\star(r)\, h(r,\theta), \tag{3.24}$$

where $v^\star$ is the radial and decreasing function given by

$$v^\star(r) = t\left(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}\right), \quad \forall r \in [0, r_0]. \tag{3.25}$$

Then, the set

$$S_{0,\tau} = \left\{(r,\theta) \in S_0 \mid v^\star(r) > \tau,\ 0 < \theta < \frac{\pi}{\alpha}\right\}$$

is a sector, and for all $p \geq 0$ we have

$$\begin{aligned} \int\limits_{S_{0,\tau}} v^{\star p} h^2 \mathrm{d}A &= \frac{\pi}{2\alpha} \int\limits_{\left\{r>0,\ t\left(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}\right)>\tau\right\}} r^{2\alpha+1} \\ &\qquad \times \left(t\left(\frac{\pi}{4\alpha(\alpha+1)}\, r^{2\alpha+2}\right)\right)^p \mathrm{d}r \\ &= \int\limits_{\{\eta>0,\quad t(\eta)>\tau\}} t^p(\eta)\mathrm{d}\eta \\ &= \int\limits_0^{\xi(\tau)} t^p(\eta)\mathrm{d}\eta \\ &= \int\limits_{D_\tau} v^p h^2 \mathrm{d}A \end{aligned} \tag{3.26}$$

for all $\tau \in [0, \overline{v}]$.

Lemma 3.7. *Choose c in (3.21) such that $R(0) = v^\star(0)$. Then*

$$z(r,\theta) \leq u^\star(r,\theta), \quad \forall\ (r,\theta) \in S_\lambda. \tag{3.27}$$

Proof. To prove this lemma we must introduce the following remark:

Remark 3.8. we have

$$\xi_0 \geq \zeta_0. \tag{3.28}$$

Indeed, let λ_0 be the lowest eigenvalue of the eigenvalue problem for S_0. By the Payne–Weinberger inequality, we have

$$\lambda_0 \leq \lambda. \tag{3.29}$$

Let r_1 be the radius of S_λ; then

$$r_1 = \frac{j_{\alpha,1}}{\sqrt{\lambda}}, \quad \text{and} \quad r_0 = \frac{j_{\alpha,1}}{\sqrt{\lambda_0}}.$$

From (3.29), we have $r_1 \leq r_0$, and, $S_\lambda \subset S_0$, then $\xi_0 \geq \zeta_0$.

We distinguish two cases:

If $\xi_0 = \zeta_0$. From the fact that D and S_λ have the same Dirichlet first eigenvalue and the Payne–Weinberger Theorem, we have $D = S_\lambda = S_0$. Now, using the fact that $\Delta h = 0$ and the divergence theorem, we have

$$\int_{S_0} |\nabla u^\star|^2 \mathrm{d}A = \int_{S_0} |\nabla v^\star|^2 h^2 \mathrm{d}A, \tag{3.30}$$

and

$$\begin{aligned}
\int_{S_0} |\nabla v^\star|^2 h^2 \mathrm{d}A &= \int_{S_0} \left|\nabla t\left(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}\right)\right|^2 h^2(r,\theta) r \mathrm{d}r \mathrm{d}\theta \\
&= \left(\frac{\pi}{2\alpha}\right)^3 \int_0^{r_0} r^{6\alpha+3} \left(t'\left(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}\right)\right)^2 \mathrm{d}r \\
&= \left(\frac{\pi}{2\alpha}\right)^{\frac{1}{\alpha+1}} (2\alpha+2)^{\frac{2\alpha+1}{\alpha+1}} \int_0^{\xi_0} \xi^{\frac{2\alpha+1}{\alpha+1}} \left(t'(\xi)\right)^2 \mathrm{d}\xi \\
&\leq \lambda \int_0^{\xi_0} (-t'(\xi)) \int_0^{\xi} t(\eta) \mathrm{d}\eta \mathrm{d}\xi \\
&= \lambda \int_0^{\xi_0} (t(\xi))^2 \mathrm{d}\xi \\
&= \lambda \int_{S_0} u^{\star 2} \mathrm{d}A.
\end{aligned} \tag{3.31}$$

Here we used (3.19), $t(\xi_0) = 0$, and integration by parts in going from the first equalities to the last ones. Thus

$$\frac{\int_{S_0} |\nabla u^\star|^2 \mathrm{d}A}{\int_{S_0} u^{\star 2} \mathrm{d}A} \leq \lambda. \tag{3.32}$$

As λ is also the minimum of the Rayleigh quotient on S_0, it follows that this minimum is achieved for $u^\star$, and so $u^\star$ is indeed the eigenfunction associated with λ on S_0. From equality (3.26), we have $u^\star = u$ and $z = z^\star$. Now, the fact that $u^\star$ and z are first Dirichlet eigenfunctions in S_0 implies the existence of $c' > 0$ such that $R(r) = c' v^\star(r), \quad \forall r \in (0, r_0)$. Finally, the hypothesis of the lemma gives $c' = 1$ and $z = u = u^\star$.

If $\xi_0 > \zeta_0$**.** We have $t(\zeta_0) > 0$ while $s(\zeta_0) = 0$. Now, by this and the fact that

$$s(0) = R(0) = v^\star(0) = \operatorname{ess\,sup} v = t(0), \tag{3.33}$$

we can find a constant $\kappa \geq 1$ such that

$$\kappa\, t(\zeta) \geq s(\zeta) \quad \forall \zeta \in [0, \zeta_0]. \tag{3.34}$$

Let c'' be the constant defined by

$$c'' = \inf\{\kappa \geq 1\,; \quad \kappa\, t(\zeta) \geq s(\zeta), \quad \forall \zeta \in [0, \zeta_0]\} \tag{3.35}$$

Then by the definition of c'', we can find $\zeta_1 \in [0, \zeta_0)$ such that $c''\, t(\zeta_1) = s(\zeta_1)$.

We define now the function g by

$$g(\zeta) = \begin{cases} c\text{''}t(\zeta); & \text{if } \zeta \in [0, \zeta_1] \\ s(\zeta); & \text{if } \zeta \in [\zeta_1, \zeta_0]. \end{cases}$$

The properties of t and s imply that g is monotonically decreasing and $g(\zeta_0) = 0$. Further, by virtue of (3.19) and (3.22), we easily see that

$$-\, g'(\zeta) \leq \lambda\, 4^{-\frac{\alpha}{\alpha+1}} \left(\frac{\pi}{\alpha}\right)^{-\frac{1}{\alpha+1}} (\alpha+1)^{-\frac{2\alpha+1}{\alpha+1}} \frac{\int_0^\xi g(\eta)\mathrm{d}\eta}{\xi^{\frac{2\alpha+1}{\alpha+1}}}, \tag{3.36}$$

for almost all $\zeta \in [0, \zeta_0]$. Now, let $\mathfrak{g}$ defined in S_λ by

$$\mathfrak{g}(r,\theta) = g\left(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}\right) h(r,\theta), \tag{3.37}$$

then, $\mathfrak{g}$ is an admissible function for the Rayleigh quotient on S_λ. From this we proceed exactly as in the proof of the inequality (3.31) and we get

$$\frac{\int_{S_\lambda} |\nabla \mathfrak{g}|^2 \mathrm{d}A}{\int_{S_\lambda} \mathfrak{g}^2 \mathrm{d}A} \leq \lambda, \tag{3.38}$$

and by the definition of S_λ it follows that $\mathfrak{g}$ is an eigenfunction associated with λ on S_λ. Then $\mathfrak{g}$ is a multiple of z and so, from the definition of $\mathfrak{g}$ (or g), it follows $c''\, t(\zeta) = s(\zeta)$ for $0 \leq \zeta \leq \zeta_1$. Since $t(0) = s(0)$, then $c'' = 1$ and $t(\zeta) \geq s(\zeta)$ for all $0 \leq \zeta \leq \zeta_0$. which proves the lemma. $\square$

Theorem 3.9 (Chiti Comparison Lemma, [13]). *For $p > 0$, let c be chosen in* (3.21) *such that*

$$\int_D v^p h^2 \mathrm{d}A = \int_{S_\lambda} R^p h^2 \mathrm{d}A, \tag{3.39}$$

and as before $r_1 = \frac{j_{\alpha,1}}{\sqrt{\lambda}}$. *Then, there exists* $r_2 \in (0, r_1)$ *such that*

$$u^\star(r,\theta) \le z(r,\theta), \quad \forall (r,\theta) \in (0, r_2] \times \left(0, \frac{\pi}{\alpha}\right); \tag{3.40}$$

$$u^\star(r,\theta) \ge z(r,\theta), \quad \forall (r,\theta) \in [r_2, r_1] \times \left(0, \frac{\pi}{\alpha}\right). \tag{3.41}$$

Remark 3.10. By virtue of (3.26), and a similar statement for z and s, the normalization condition (3.39) is equivalent to

$$\int_0^{\xi_0} t^p(\xi)\mathrm{d}\xi = \int_0^{\zeta_0} s^p(\zeta)\mathrm{d}\zeta. \tag{3.42}$$

Since the functions t and s are nonnegative, and $\zeta_0 \le \xi_0$ (see Remark 3.8), it is then clear that

$$\int_0^{\zeta_0} t^p(\zeta)\mathrm{d}\zeta \le \int_0^{\zeta_0} s^p(\zeta)\mathrm{d}\zeta. \tag{3.43}$$

Proof. We will first prove that $s(0) \ge t(0)$.

Assume that $s(0) < t(0)$.

In this case, $\exists\, \kappa > 1$, such that $\kappa\, s(0) = t(0)$. By Lemma 3.7, it then follows that

$$\kappa\, s(\zeta) \le t(\zeta) \quad \forall \zeta \in [0, \zeta_0]. \tag{3.44}$$

Therefore,

$$\kappa^p \int_0^{\zeta_0} s^p(\zeta)\mathrm{d}\zeta \le \int_0^{\zeta_0} t^p(\zeta)\mathrm{d}\zeta.$$

Combining this inequality with (3.43) leads to $\kappa^p \le 1$, which is a contradiction.

Suppose now that $s(0) = t(0)$.

From (3.42) and Lemma 3.7 we obtain

$$\int_0^{\xi_0} t^p(\zeta)\mathrm{d}\zeta = \int_0^{\zeta_0} s^p(\zeta)\mathrm{d}\zeta \le \int_0^{\zeta_0} t^p(\zeta)\mathrm{d}\zeta. \tag{3.45}$$

This means $\int_{\zeta_0}^{\xi_0} t^p(\zeta)\mathrm{d}\zeta = 0$, and since $t > 0$ in $(0, \xi_0)$, we have $\xi_0 = \zeta_0$. Then $z = u^\star$, and the statements of the theorem are evident.

Now, we treat the case $s(0) > t(0)$.

In this case $\zeta_0 < \xi_0$ (this is evident from the proof of Lemma 3.7). Therefore, $s(\zeta_0) = 0$ and $t(\zeta_0) > 0$. Now, by the continuity of t and s, we see that $s(\zeta) > t(\zeta)$ in a neighborhood of 0, and there exists $\zeta_1 \in (0, \zeta_0)$ such that $s(\zeta_1) = t(\zeta_1)$. Choose ζ_1 to be the largest such number with the additional property that $t(\zeta) \le s(\zeta)$ for all $\zeta \in [0, \zeta_1]$. By the definition of ζ_1, there is an interval immediately to the right of ζ_1 on which $t(\zeta) > s(\zeta)$. We will now show that $t(\zeta) > s(\zeta)$ for all $\zeta \in (\zeta_1, \zeta_0]$. If not, there exists $\zeta_2 \in (\zeta_1, \zeta_0)$ such that

$t(\zeta_2) = s(\zeta_2)$ and $t(\zeta) > s(\zeta)$ for all $\zeta \in (\zeta_1, \zeta_2)$. In this case, we can define the function

$$\varphi(\zeta) = \begin{cases} s(\zeta), & \text{for } \zeta \in [0, \zeta_1] \cup [\zeta_2, \zeta_0], \\ t(\zeta), & \text{for } \zeta \in [\zeta_1, \zeta_2]. \end{cases}$$

It follows from (3.19) and (3.22) that φ satisfies

$$-\varphi'(\zeta) \leq \lambda\, 4^{-\frac{\alpha}{\alpha+1}} \left(\frac{\pi}{\alpha}\right)^{-\frac{1}{\alpha+1}} (\alpha+1)^{-\frac{2\alpha+1}{\alpha+1}} \frac{\int_0^\zeta \varphi(\eta)\mathrm{d}\eta}{\zeta^{\frac{2\alpha+1}{\alpha+1}}}. \tag{3.46}$$

From φ define the function in S_λ by

$$\Phi(r,\theta) = \varphi\left(\frac{\pi}{4\alpha(\alpha+1)} r^{2\alpha+2}\right) h(r,\theta). \tag{3.47}$$

Then Φ is an admissible function for the Rayleigh quotient on S_λ. From this and proceeding exactly as in the proof of the inequality (3.38), we have

$$\frac{\int_{S_\lambda} |\nabla\Phi|^2 \mathrm{d}A}{\int_{S_\lambda} \Phi^2 \mathrm{d}A} \leq \lambda. \tag{3.48}$$

It will follow that the Rayleigh quotient of Φ is equal to λ and hence that Φ is an eigenfunction for λ, Consequently, $t = s$ and so $t(\zeta) = s(\zeta)$ in $[\zeta_1, \zeta_2]$ contradicting the maximality of ζ_1. The statements of our theorem follow. □

3.1. Proof of Theorem 3.1

For $p > 0$, we choose c in (3.21) so that (3.39) is satisfied. This means

$$\int_0^{\xi_0} t^p(\xi)\mathrm{d}\xi = \int_0^{\zeta_0} s^p(\xi)\mathrm{d}\xi \tag{3.49}$$

as we pointed out in Remark 3.10.

Now, if we extend the function s by zero in $[\zeta_0, \xi_0]$, we obtain

$$\int_0^{\xi} t^p(\eta)\mathrm{d}\eta \leq \int_0^{\xi} s^p(\eta)\mathrm{d}\eta, \quad \forall \xi \in [0, \xi_0]. \tag{3.50}$$

To see (3.50), we let $\zeta_1 = \frac{\pi}{4\alpha(\alpha+1)} r_2^{2\alpha+2}$. We then note that Theorem 3.9 implies the following:

If $\xi \in [0, \zeta_1]$, then

$$t(\eta) \leq s(\eta) \quad \forall \eta \in [0, \xi],$$

and so

$$\int_0^{\xi} t^p(\eta)\mathrm{d}\eta \leq \int_0^{\xi} s^p(\eta)\mathrm{d}\eta.$$

If $\xi \in [\zeta_1, \xi_0]$, then

$$\int_0^{\xi} t^p(\eta)\mathrm{d}\eta = \int_0^{\xi_0} t^p(\eta)\mathrm{d}\eta - \int_{\xi}^{\xi_0} t^p(\eta)\mathrm{d}\eta$$
$$\leq \int_0^{\xi_0} s^p(\eta)\mathrm{d}\eta - \int_{\xi}^{\xi_0} s^p(\eta)\mathrm{d}\eta$$
$$= \int_0^{\xi} s^p(\eta)\mathrm{d}\eta.$$

We complete the argument using the following result:

Lemma 3.11. *Let M, p, q be real numbers such that $0 < p \leq q$, $M > 0$; let f, g be real functions in $L^q([0, M])$,. If the decreasing rearrangements f and g satisfy the inequality*

$$\int_0^s (f^*)^p \, \mathrm{d}t \leq \int_0^s (g^*)^p \, \mathrm{d}t, \quad \forall\, s \in [0, M],$$

then

$$\int_0^M f^q \mathrm{d}t \leq \int_0^M g^q \mathrm{d}t.$$

Proof. The result is a corollary of a theorem of Hardy, Littlewood and Pólya proved in [18] (Theorem 10, p. 152). □

From this, it is clear, for $q \geq p$, that

$$\int_0^{\xi_0} t^q(\eta)\mathrm{d}\eta \leq \int_0^{\xi_0} s^q(\eta)\mathrm{d}\eta = \int_0^{\zeta_0} s^q(\eta)\mathrm{d}\eta. \tag{3.51}$$

Using this and equality (3.49), we see

$$\left(\int_D u^q\, h^{2-q}\mathrm{d}A\right)^{\frac{1}{q}} \leq K(p, q, \lambda, \alpha) \left(\int_D u^p\, h^{2-p}\mathrm{d}A\right)^{\frac{1}{p}}, \tag{3.52}$$

with

$$K(p, q, \lambda, \alpha) = \frac{\left(\int_{S_\lambda} c^q\, R^q h^{2-q}\mathrm{d}A\right)^{\frac{1}{q}}}{\left(\int_{S_\lambda} c^p R^p\, h^{2-p}\mathrm{d}A\right)^{\frac{1}{p}}}$$
$$= \frac{\left(\int_0^{\frac{\pi}{\alpha}} \int_0^{r_0} r^{(2-q)\alpha+1} J_\alpha^q(\sqrt{\lambda} r) \sin^2(\alpha\theta)\mathrm{d}r\mathrm{d}\theta\right)^{\frac{1}{q}}}{\left(\int_0^{\frac{\pi}{\alpha}} \int_0^{r_0} r^{(2-p)\alpha+1} J_\alpha^p(\sqrt{\lambda} r) \sin^2(\alpha\theta)\mathrm{d}r\mathrm{d}\theta\right)^{\frac{1}{p}}}$$

$$= \left(\frac{\pi}{2\alpha}\right)^{\frac{p-q}{pq}} \lambda^{(\alpha+1)\frac{q-p}{pq}} \frac{\left(\int_0^{j_{\alpha,1}} r^{(2-q)\alpha+1} J_\alpha^q(r)\mathrm{d}r\right)^{\frac{1}{q}}}{\left(\int_0^{j_{\alpha,1}} r^{(2-p)\alpha+1} J_\alpha^p(r)\mathrm{d}r\right)^{\frac{1}{p}}}.$$

The Proof of Theorem 3.1 is now complete.

4. Sharp Estimates for the Dirichlet Problem of a Wedge-Like Membrane

In this part we consider the following class of degenerate elliptic equations:

$$\mathcal{P}_3 : \begin{cases} -div(h^k \nabla u) = h^k f & \text{in } D \\ u = 0 & \text{on } \partial D \cap \mathcal{W}, \end{cases}$$

where $k > 0$, and $h(r, \theta) = r^\alpha \sin \alpha\theta$, as defined earlier. As in the previous section, $D \subset \mathcal{W}$ is a bounded domain with piecewise smooth boundary. Finally, we let f be a smooth function defined in D, and $f^\star$ denote its weighted symmetrization (as defined below in Sect. 4.2).

We also introduce the measure $\mathrm{d}\mu$ defined by

$$\mathrm{d}\mu = h^k \,\mathrm{d}A = r^{\alpha k+1} \left(\sin \alpha\theta\right)^k \mathrm{d}r\mathrm{d}\theta. \tag{4.1}$$

We let $\mu(D) = \int_D \mathrm{d}\mu$, and S_0 be the sector such that $\mu(D) = \mu(S_0)$, with r_0 denoting the radius of S_0. We point out similar treatments for different measures $\mathrm{d}\mu$ in other works [7,8,10,11,31,52].

Theorem 4.1. *Let u be the solution to problem $\mathcal{P}_3$ and let v be the function defined by*

$$v(r, \theta) = v^\star(r) = \int_r^{r_0} \left(\int_0^\delta f^\star(\rho)\rho^{\alpha k+1} \,\mathrm{d}\rho \right) \delta^{-(\alpha k+1)} \,\mathrm{d}\delta, \tag{4.2}$$

which is the weak solution to the symmetrized problem

$$\mathcal{P}_4 : \begin{cases} -div(h^k \nabla v) = h^k f^\star & \text{in } S_0 \\ v = 0 & \text{on } \partial S_0 \cap \mathcal{W}. \end{cases}$$

Then $u^\star$, the weighted symmetrization of u satisfies

$$u^\star \le v \quad a.e \quad in \;\; S_0. \tag{4.3}$$

Moreover,

$$\int_D |\nabla u|^q \mathrm{d}\mu \le \int_{S_0} |\nabla v|^q \mathrm{d}\mu, \qquad 0 < q \le 2. \tag{4.4}$$

We note that the function v, the solution to the symmetrized problem $\mathcal{P}_4$, which is also symmetric, is radial, and thus is independent of θ.

Theorem 4.2. *Let u be the solution of problem $\mathcal{P}_3$. Then*

(1) *For* $p > 1 + \frac{\alpha k}{2}$,

$$\text{ess sup}\,|u| \leq \mu(D)^{\frac{2}{\alpha k+2}-\frac{1}{p}} \frac{p(\alpha k+2)}{C(\alpha,k)^2\,(2(p-1)-\alpha k)} \left(\int_D |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}}$$

(2) *For* $1 < p < \frac{2(\alpha k+2)}{\alpha k+4}$, *and* $q = \frac{p(\alpha k+2)}{\alpha k+2-p}$, *one has*

$$\int_D |\nabla u|^q \mathrm{d}\mu \leq \mathcal{A}\, C^{-q}(\alpha,k) \left(\int_D |f|^p \mathrm{d}\mu\right)^{\frac{q}{p}},$$

where

$$\mathcal{A} = \frac{p}{q(p-1)} \left(\frac{\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{q}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{q}{p}-1},$$

$$C(\alpha,k) = \left(\frac{(\alpha\,k+2)^{\alpha\,k+1}}{\alpha} B\left(\frac{1}{2},\frac{k+1}{2}\right)\right)^{1/(\alpha k+2)} \tag{4.5}$$

and B denoting the Euler Beta function.

Remark 4.3. Maderna and Salsa proved versions of Theorem 4.1 and Theorem 4.2 corresponding to the $k > 0$, $\alpha = 1$ case (compare with i) and ii) in Theorem 3 of [31]). We also note that our theorems have the same Weinstein space interpretation in $(k\alpha + 2)$ dimension of (i) and (ii) in Theorem 2 of [51]. In fact, these observations were the main motivation for our theorems.

After some geometric preparation in Sect. 4.1, we define weighted rearrangement and prove necessary propositions in Sect. 4.2 and then finally prove these two theorems in Sect. 4.3.

4.1. A Geometric Inequality

The following result generalizes earlier work by Bandle and Payne–Weinberger:

Proposition 4.4. *let $D \subset \mathcal{W}$ be a bounded domain with a piecewise smooth boundary. Then, for any nonnegative number k, we have*

$$\int_{\partial D} h^k(r,\theta)\sqrt{\mathrm{d}r^2 + r^2\mathrm{d}\theta^2} \geq C(\alpha,k) \left(\int_D h^k(r,\theta) r\mathrm{d}r\mathrm{d}\theta\right)^{(\alpha k+1)/(\alpha k+2)} \tag{4.6}$$

with $C(\alpha,k)$ as defined in (4.5). Equality holds if and only if D is a circular sector of angle $\frac{\pi}{\alpha}$.

Remark 4.5. The special case $k = 0$, $\alpha \geq 1$ appears in the first few pages of Bandle's book in the context of α-symmetrization (see Theo. 1.1. of [5]). Lions and Pacella [29] extended this $k = 0$ case to solid angles in higher dimensions and proved it using the Brunn-Minkowski method. Using a particular form of the Szegő Lemma [49], Payne and Weinberger offer a complete proof for

$k = 2$ and $\alpha \geq 1$ in [39] (this is Lemma 3.5 above). An independent, entirely different proof is offered in [16]; see also Remark 2 in [44] where the $k = 2$ case for convex cones in higher dimensions appears, and the discussion in [45]. The proposition follows from the $\alpha = 1$, $k \geq 0$ case proved by Maderna-Salsa (see Lemma 4.6).

Proof. To prove the isoperimetric result of the proposition, we will need to use the following lemma:

Lemma 4.6 (Maderna-Salsa [31]). *Let $\widetilde{D}$ be a bounded domain in the upper half-plane $\mathbb{R}^2_+$ with piecewise smooth boundary, and $k \geq 0$. Then*

$$\int_{\partial \widetilde{D}} y^k \, d\widetilde{\sigma} \geq C(1,k) \left(\int_{\widetilde{D}} y^k \mathrm{d}x\mathrm{d}y \right)^{\frac{k+1}{k+2}}$$

where $\widetilde{\sigma}$ is the parameter of arclength on $\partial \widetilde{D}$ and $C(1,k)$ is the expression in (4.5) when $\alpha = 1$. The case of equality holds if and only if D is a semicircle centered on the x-axis.

Next, we proceed as in Payne–Weinberger [39] (see also [44]). Let Υ be the map from $\mathcal{W}$ into $\mathbb{R}^2_+$ defined by

$$\Upsilon(r,\theta) = r^{\frac{\alpha k+1}{k+1}} (\cos(\alpha\theta), \sin(\alpha\theta)).$$

From the fact that

$$\det D\Upsilon = \alpha \frac{\alpha k + 1}{k+1} r^{\frac{k(2\alpha-1)+1}{k+1}} > 0,$$

we know that the map Υ is a diffeomorphism. Now we estimate the effect of Υ on the arclength and surface elements. With $\mathrm{d}\sigma = \sqrt{\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2}$ denoting the arclength element in D as before, we let $\mathrm{d}\widetilde{\sigma} = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$ be the arclength element in the image domain $\widetilde{D}$. Since $\alpha \geq 1$, a direct computation shows that

$$\begin{aligned} \mathrm{d}\widetilde{\sigma}^2 = \mathrm{d}x^2 + \mathrm{d}y^2 &= \left(\frac{\alpha k + 1}{k+1} \right)^2 r^{\frac{2(\alpha k - k)}{k+1}} \mathrm{d}r^2 + \alpha^2 r^{\frac{2(\alpha k+1)}{k+1}} \mathrm{d}\theta^2 \\ &\leq \alpha^2 r^{\frac{2(\alpha k - k)}{k+1}} \left(\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 \right). \end{aligned}$$

Therefore,

$$\begin{aligned} y^k \mathrm{d}\widetilde{\sigma} &\leq \alpha r^{\frac{k(\alpha k+1)}{k+1}} r^{\frac{\alpha k - k}{k+1}} \sin^k(\alpha\theta) \mathrm{d}\sigma \\ &= \alpha h^k(r,\theta) \mathrm{d}\sigma. \end{aligned} \tag{4.7}$$

Now, applying the Maderna-Salsa Lemma 4.6, we get

$$\begin{aligned} \int_{\partial D} h^k(r,\theta) \mathrm{d}\sigma &\geq \alpha^{-1} \int_{\partial \widetilde{D}} y^k \mathrm{d}\widetilde{\sigma} \\ &\geq \alpha^{-1} C(1,k) \left(\int_{\widetilde{D}} y^k \mathrm{d}x\mathrm{d}y \right)^{\frac{k+1}{k+2}} \end{aligned}$$

$$= \alpha^{-1} C(1,k) \left(\int\limits_D r^{\frac{k(\alpha k+1)}{k+1}} \sin^k(\alpha\theta) \Big| \det(D\Upsilon) \Big| \mathrm{d}r \mathrm{d}\theta \right)^{\frac{k+1}{k+2}}$$

$$= C(1,k)\, \alpha^{-\frac{1}{k+2}} \left(\frac{\alpha k+1}{k+1} \right)^{\frac{k+1}{k+2}} \left(\int\limits_D r^{\frac{\alpha k^2+2\alpha k+1}{k+1}} \sin^k(\alpha\theta) \mathrm{d}r \mathrm{d}\theta \right)^{\frac{k+1}{k+2}}. \quad (4.8)$$

For θ given in $(0, \frac{\pi}{\alpha})$, we introduce the radial slice

$$D(\theta) = \{ r \geq 0; \quad (r,\theta) \in D \},$$

and let

$$f(r) = (\ell+1) r^\ell, \quad g(r) = r^m, \quad (4.9)$$

where

$$\ell = \alpha k + 1, \quad m = \frac{\alpha k - k}{(k+1)(\alpha k+2)}. \quad (4.10)$$

Using this notation, we rewrite the inner integral of inequality (4.8) to get

$$\int\limits_D r^{\frac{\alpha k^2+2\alpha k+1}{k+1}} \sin^k(\alpha\theta) \mathrm{d}r \mathrm{d}\theta = \int\limits_0^{\frac{\pi}{\alpha}} \int\limits_{D(\theta)} r^\ell (r^{\ell+1})^m \mathrm{d}r \sin^k(\alpha\theta) \mathrm{d}\theta. \quad (4.11)$$

The next necessary step is a result of Szegő which has a long history [39,43, 44,49]. The form that is most useful for us is its version in [44].

Lemma 4.7 (Szegő, [49]). *If f is nonnegative function, g is nondecreasing function and*

$$F(t) = \int\limits_0^t f(x) \mathrm{d}x, \quad G(t) = \int\limits_0^t g(x) \mathrm{d}x, \quad (4.12)$$

then for any bounded measurable set $E \subset \mathbb{R}$,

$$G\left(\int\limits_E f(x) \mathrm{d}x \right) \leq \int\limits_E g\left(F(x)\right)\, f(x) \mathrm{d}x, \quad (4.13)$$

with equality if and only if E is almost everywhere an interval of the form $[0,R]$.

Let

$$I = \int\limits_D r^{\frac{\alpha k^2+2\alpha k+1}{k+1}} \sin^k(\alpha\theta) \mathrm{d}r \mathrm{d}\theta.$$

Using Lemma 4.7, with $f(r)$ and $g(r)$ as defined by (4.9), we have

$$\begin{aligned} I &= \frac{1}{\ell+1}\int_0^{\frac{\pi}{\alpha}}\int_{D(\theta)} f(r)\, g\left(F(r)\right)\, \mathrm{d}r \sin^k(\alpha\theta)\mathrm{d}\theta \\ &\geq \frac{1}{\ell+1}\int_0^{\frac{\pi}{\alpha}} G\left(\int_{D(\theta)} f(r)\mathrm{d}r\right)\sin^k(\alpha\theta)\mathrm{d}\theta \\ &= \frac{1}{(\ell+1)(m+1)}\int_0^{\frac{\pi}{\alpha}}\left(\int_{D(\theta)} r^\ell \mathrm{d}r\right)^{m+1}\sin^k(\alpha\theta)\mathrm{d}\theta. \end{aligned} \tag{4.14}$$

Moreover, equality holds if and only if $D(\theta)$ is an interval of the form $(0, R(\theta))$ for almost every θ.

Using Hölder's inequality for the functions $f(\theta) = \int_{D(\theta)} r^\ell \mathrm{d}r \left(\sin^k(\alpha\theta)\right)^{\frac{1}{m+1}}$ and $\left(\sin^k(\alpha\theta)\right)^{\frac{m}{m+1}}$, with respective exponents $(m+1)$ and $\frac{m+1}{m}$, we obtain

$$\begin{aligned} &\left(\int_0^{\frac{\pi}{\alpha}}\left(\int_{D(\theta)} r^\ell \mathrm{d}r\right)^{m+1}\sin^k(\alpha\theta)\mathrm{d}\theta\right)^{\frac{1}{m+1}}\left(\int_0^{\frac{\pi}{\alpha}}\sin^k(\alpha\theta)\mathrm{d}\theta\right)^{\frac{m}{m+1}} \\ &\geq \int_0^{\frac{\pi}{\alpha}}\int_{D(\theta)} r^\ell \mathrm{d}r \sin^k(\alpha\theta)\mathrm{d}\theta. \end{aligned}$$

We also note that

$$\int_0^{\frac{\pi}{\alpha}}\sin^k(\alpha\theta)\mathrm{d}\theta = \frac{1}{\alpha}B\left(\frac{1}{2},\frac{k+1}{2}\right).$$

Using these last two statements and inequality (4.14), we get

$$\int_D r^{\frac{\alpha k^2+2\alpha k+1}{k+1}}\sin^k(\alpha\theta)\mathrm{d}r\mathrm{d}\theta \geq \delta\left(\int_0^{\frac{\pi}{\alpha}}\int_{D(\theta)} r^\ell \mathrm{d}r \sin^k(\alpha\theta)\mathrm{d}\theta\right)^{m+1} \tag{4.15}$$

with

$$\delta = \frac{1}{(\ell+1)(m+1)}\left(\frac{1}{\alpha}B\left(\frac{1}{2},\frac{k+1}{2}\right)\right)^{-m}.$$

In this case, equality holds in (4.15) if and only if $\int_{D(\theta)} r^\ell \mathrm{d}r$ is independent of θ. Now, plug (4.15) into inequality (4.8) to obtain

$$\int\limits_{\partial D} h^k(r,\theta)\mathrm{d}\sigma \geq C(\alpha,k)\left(\int\limits_0^{\frac{\pi}{\alpha}}\int\limits_{D(\theta)} r^\ell \mathrm{d}r \sin^k(\alpha\theta)\mathrm{d}\theta\right)^{\frac{(k+1)(m+1)}{k+2}}. \tag{4.16}$$

Here $C(\alpha,k)$ is given by (4.5). By virtue of (4.10) and the definition of $h(r,\theta)$

$$\int\limits_D h^k(r,\theta) r\mathrm{d}r\mathrm{d}\theta = \int\limits_0^{\frac{\pi}{\alpha}}\int\limits_{D(\theta)} r^\ell \mathrm{d}r \sin^k(\alpha\theta)\mathrm{d}\theta. \tag{4.17}$$

Combining (4.16) and (4.17), and simplifying the exponent, we obtain the isoperimetric result.

In the case of the perfect sector it is not difficult to check that equality is attained. As in Ratzkin's [44], the case of equality in (4.6) means equality in all the intermediate steps, in particular (4.14). Using Szegő Lemma 4.7, the radial slice $D(\theta)$ must be an interval $(0,R(\theta))$, and

$$\int\limits_{D(\theta)} r^\ell \mathrm{d}r = \frac{1}{\ell+1}R^{\ell+1}(\theta). \tag{4.18}$$

Moreover, equality in Hölder's inequality forces $\int_{D(\theta)} r^\ell \mathrm{d}r$ to be a constant function of θ. By virtue of (4.18), $R(\theta)$ must be a constant function of θ, which means the domain D must be a sector. □

4.2. Weighted Rearrangement and the Pólya–Szegő Principle for a Wedge-Like Membrane

In this part we will introduce some definitions and results about weighted rearrangement with respect to the measure $\mathrm{d}\mu$ defined by (4.1) (see, e.g., [52]). Let u be a real-valued measurable function defined in $D \subset \mathcal{W}$, and S_0 the sector defined above with $\mu(S_0) = \mu(D) = \int_D \mathrm{d}\mu$. Then the distribution function of u with respect to $\mathrm{d}\mu$ is m_u defined by

$$m_u(t) = \mu\left(\{(r,\theta) \in D;\ |u(r,\theta)| > t\}\right), \quad \forall t \in [0, \operatorname{ess\,sup}|u|]. \tag{4.19}$$

The decreasing rearrangement with respect to $\mathrm{d}\mu$ of u is the function

$$u^* : [0,\mu(D)] \to [0,+\infty)$$

defined by

$$u^*(0) = \operatorname{ess\,sup}|u|,$$
$$u^*(s) = \inf\{t \geq 0;\ m_u(t) < s\}, \quad \forall s \in (0,\mu(D)].$$

The weighted rearrangement of u is the function $u^\star$ from the sector S_0 into $[0,+\infty)$ defined by

$$u^\star(r,\theta) = u^*\left(\beta(\alpha,k)\, r^{\alpha k+2}\right), \tag{4.20}$$

where

$$\beta(\alpha,k)=\frac{1}{\alpha(\alpha k+2)}B\left(\frac{1}{2},\frac{k+1}{2}\right). \tag{4.21}$$

The function $u^\star$ is a radial and nonincreasing and its level sets are sectors centered at the origin whose weighted measure is $m_u(t)$. We will abuse notation by letting $u^\star(r,\theta)$ be $u^\star(r)$. Using now the Cavalieri principle, we have

$$\int_D |u|^p \mathrm{d}\mu=\int_{S_0} u^{\star p}\mathrm{d}\mu, \quad \forall p\in[1,+\infty). \tag{4.22}$$

By a solution to problem $\mathcal{P}_3$ we mean a measurable function u whose weak gradient is square integrable in D with respect to the measure $\mathrm{d}\mu$ and which satisfies the boundary condition in the following sense: there exists a sequence of functions $u_n\in C^1(\overline{D})$ such that $u_n(r,\theta)=0$ on $\partial D\cap\mathcal{W}$ and

$$\lim_{n\to+\infty}\int_D |\nabla(u-u_n)|^2\mathrm{d}\mu+\int_D |u-u_n|^2\mathrm{d}\mu=0. \tag{4.23}$$

Moreover, u satisfies the equality

$$\int_D \langle\nabla u,\nabla\psi\rangle\mathrm{d}\mu=\int_D f\psi\mathrm{d}\mu, \tag{4.24}$$

for every $\psi\in C^1(\overline{D})$ such that $\psi(r,\theta)=0$ on $\partial D\cap\mathcal{W}$. One can relax the conditions of f by requiring it to be in $L^2(D,\mathrm{d}\mu)$.

We now introduce the space $W_k(D,\mathrm{d}\mu)$ which is the set of measurable functions u which satisfy the following conditions:

(i) $\int_D |\nabla u|^2\mathrm{d}\mu+\int_D |u|^2\mathrm{d}\mu<+\infty$

(ii) There exists a sequence of functions $u_n\in C^1(\overline{D})$ such that $u_n(r,\theta)=0$ on $\partial D\cap\mathcal{W}$ and

$$\lim_{n\to+\infty}\int_D |\nabla(u-u_n)|^2\mathrm{d}\mu+\int_D |u-u_n|^2\mathrm{d}\mu=0. \tag{4.25}$$

The definition of $W_k(D,\mathrm{d}\mu)$ is motivated by similar consideration in [10,11, 31,52] (see also [7,8,51]). $W_k(D,\mathrm{d}\mu)$ is a Hilbert space with inner product

$$\langle f,g\rangle=\int_D (\nabla f\cdot\nabla g+fg)\ \mathrm{d}\mu.$$

We will now prove the following weighted version of the Pólya–Szegő inequality for the Dirichlet integral, for wedge-like domains:.

Proposition 4.8 (Weighted Pólya–Szegő). *Let u be a nonnegative function in $W_k(D,\mathrm{d}\mu)$. Then $u^\star\in W_k(S_0,\mathrm{d}\mu)$, and*

$$\int_D |\nabla u|^2\mathrm{d}\mu\geq\int_{S_0} |\nabla u^\star|^2\mathrm{d}\mu \tag{4.26}$$

Proof. We proceed as in [10,11]. For brevity's sake, we will list the key ingredients. For $u \in C^1(\overline{D})$ with $u = 0$ on the $\partial D \cap \mathcal{W}$, we can proceed as in [52] and obtain the weighted Pólya–Szegő isoperimetric inequality of this lemma.

Now let $u \in W_k(D, \mathrm{d}\mu)$ and (u_n) be a sequence of functions verifying (ii); then from a nonexpansivity of the rearrangement [24], i.e.,

$$\int\limits_{S_0} |u_n^\star - u^\star|^2 \,\mathrm{d}\mu \leq \int\limits_{D} |u_n - u|^2 \,\mathrm{d}\mu$$

we can deduce that

$$u_n^\star \longrightarrow u^\star \quad \text{in} \quad L^2(S_0, \mathrm{d}\mu). \tag{4.27}$$

Since

$$\int\limits_{D} |\nabla u_n|^2 \mathrm{d}\mu \geq \int\limits_{S_0} |\nabla u_n^\star|^2 \mathrm{d}\mu,$$

the sequence $(u_n^\star)$ is bounded in $W_k(S_0, \mathrm{d}\mu)$. Thus the sequence $(u_n^\star)$ has a weakly convergent subsequence in this space. Using (4.27), this subsequence $u_{n'}^\star$ converges weakly to $u^\star$ in $W_k(S_0, \mathrm{d}\mu)$. Now, the weak lower semi-continuity of the norm and equality (4.22) completes the proof. □

Corollary 4.9 (Weighted Poincaré Inequality). *For any function u belonging to $W_k(D, \mathrm{d}\mu)$, we have*

$$\int\limits_{D} |u|^2 \mathrm{d}\mu \leq C \int\limits_{D} |\nabla u|^2 \mathrm{d}\mu, \tag{4.28}$$

where C is a positive constant depending only on $\mu(D)$.

Proof. From (4.22), the Pólya–Szegő inequality and Theorem 3 in Mazya's book [33] (page 47), we have

$$\frac{\int_D |\nabla u|^2 \mathrm{d}\mu}{\int_D |u|^2 \mathrm{d}\mu} \geq \frac{\int_{S_0} |\nabla u^\star|^2 \mathrm{d}\mu}{\int_{S_0} |u^\star|^2 \mathrm{d}\mu} = C^2(\alpha, k) \frac{\int_0^{\mu(D)} \left(\frac{\partial u^*}{\partial s}\right)^2 s^{\frac{2(\alpha k+1)}{\alpha k+2}} \mathrm{d}s}{\int_0^{\mu(D)} u^{*2} \mathrm{d}s} \geq C \tag{4.29}$$

where C is a positive constant depending on $\mu(D), k$ and α. □

By Corollary 4.9 and the Lax–Milgram theorem, we easily deduce the existence and uniqueness of the solutions to problems $\mathcal{P}_3$ and $\mathcal{P}_4$. We can also deduce from the Poincaré-type inequality (4.28) that the functional space $W_k(D, \mathrm{d}\mu)$ can be equivalently equipped with the norm

$$\| u \|_{W_k(D, \mathrm{d}\mu)} = \int\limits_{D} |\nabla u|^2 \mathrm{d}\mu. \tag{4.30}$$

Results in similar settings, chiefly influenced by [52], appear in [7,8,10,11,31].

4.3. Sharp Estimates for Solution of Problem $\mathcal{P}_3$

In this part we will prove the main theorems of this section.

4.3.1. Proof of Theorem 4.1. This proof is inspired by [51]. In the weak formulation (4.24) of the problem $\mathcal{P}_3$, choose the test function ψ defined by

$$\psi(r,\theta) = \begin{cases} (|u(r,\theta)| - t)\ \mathrm{sign}(u), & \text{if } |u(r,\theta)| > t \\ 0, & \text{otherwise}, \end{cases} \tag{4.31}$$

where $0 \le t < \operatorname{ess\,sup}|u|$. Plugging (4.31) into (4.24) we get

$$\int_{|u|>t} |\nabla u|^2 \mathrm{d}\mu = \int_{|u|>t} (|u| - t)\,\mathrm{sign}(u)\, f\, \mathrm{d}\mu. \tag{4.32}$$

Now, let Φ be the decreasing function of t defined by

$$\Phi(t) = \int_{|u|>t} |\nabla u|^2 \mathrm{d}\mu. \tag{4.33}$$

Then, for $\varepsilon > 0$, we have

$$\frac{\Phi(t) - \Phi(t+\varepsilon)}{\varepsilon} = \int_{|u|>t+\varepsilon} \mathrm{sign}(u)\, f\, \mathrm{d}\mu + \int_{t<|u|\le t+\varepsilon} \left(\frac{|u| - t}{\varepsilon}\right) \mathrm{sign}(u)\, f\, \mathrm{d}\mu.$$

Letting ε go to zero, we obtain, for the right derivative of $\Phi(t)$,

$$-\Phi'_+(t) = \int_{|u|>t} \mathrm{sign}(u)\, f\, \mathrm{d}\mu \quad \text{a.e.} \quad t > 0. \tag{4.34}$$

The same computation gives the same equality for the left derivative of $\Phi(t)$. Therefore,

$$0 \le -\Phi'(t) \le \int_{|u|>t} |f| \mathrm{d}\mu. \tag{4.35}$$

We next use the Cauchy–Schwarz inequality

$$\left(\frac{1}{\varepsilon} \int_{t<|u|\le t+\varepsilon} |\nabla u| \mathrm{d}\mu\right)^2 \le \left(\frac{1}{\varepsilon} \int_{t<|u|\le t+\varepsilon} |\nabla u|^2 \mathrm{d}\mu\right) \left(\frac{1}{\varepsilon} \int_{t<|u|\le t+\varepsilon} \mathrm{d}\mu\right). \tag{4.36}$$

Thus, letting $\varepsilon \to 0$ and using (4.35), we obtain

$$\left(-\frac{d}{dt} \int_{|u|>t} |\nabla u| \mathrm{d}\mu\right)^2 \le \left(\int_{|u|>t} |f| \mathrm{d}\mu\right) (-m'_u(t)). \tag{4.37}$$

Now, using the Hardy–Littlewood theorem, we get

$$\left(-\frac{d}{dt} \int_{|u|>t} |\nabla u| \mathrm{d}\mu\right)^2 \le \left(\int_0^{m_u(t)} f^*(s) \mathrm{d}s\right) (-m'_u(t)). \tag{4.38}$$

From the co-area formula, we have

$$-\frac{d}{\mathrm{d}t}\int\limits_{|u|>t}|\nabla u|\mathrm{d}\mu=\int\limits_{\partial\{|u|>t\}}h^k\mathrm{d}\sigma,\quad \text{a.e.}\quad t>0. \tag{4.39}$$

But by our isoperimetric inequality (4.6), we have

$$\int\limits_{\partial\{|u|>t\}}h^k\mathrm{d}\sigma\geq\int\limits_{\partial\{u^\star>t\}}h^k\mathrm{d}\sigma=C(\alpha,k)\,(m_u(t))^{(\alpha k+1)/(\alpha k+2)}. \tag{4.40}$$

Combining (4.39) and (4.40), we get

$$1\leq C(\alpha,k)^{-2}\,(-m_u'(t))\;(m_u(t))^{\frac{-2(\alpha k+1)}{\alpha k+2}}\int\limits_0^{m_u(t)}f^*(s)\mathrm{d}s, \tag{4.41}$$

for almost every t in $(0,\text{ess sup}\,|u|)$. Integrating this inequality from 0 to t, we get

$$t\leq C(\alpha,k)^{-2}\int\limits_{m_u(t)}^{\mu(D)}\xi^{\frac{-2(\alpha k+1)}{\alpha k+2}}\int\limits_0^{\xi}f^*(s)\mathrm{d}s\mathrm{d}\xi. \tag{4.42}$$

Now, for $s\in(0,\mu(D))$, let $t=u^*(s)-\gamma>0$ where $\gamma>0$. By the definition of the rearrangement u^*, we have $m_u(t)\geq s$. Using this in (4.42) and letting $\gamma\to 0$, we get

$$u^*(s)\leq C(\alpha,k)^{-2}\int\limits_{s}^{\mu(D)}\xi^{\frac{-2(\alpha k+1)}{\alpha k+2}}\int\limits_0^{\xi}f^*(s')\mathrm{d}s'\mathrm{d}\xi. \tag{4.43}$$

Recalling that $u^\star(r)=u^*\left(\beta(\alpha,k)r^{\alpha k+2}\right)$ and using the change of variable $\delta=\left(\frac{1}{\beta(\alpha,k)}\xi\right)^{\frac{1}{\alpha k+2}}$, we get

$$u^\star(r)\leq(\alpha k+2)\,C(\alpha,k)^{-2}\beta(\alpha,k)^{\frac{-\alpha k}{\alpha k+2}}\int\limits_r^{r_0}\delta^{-(\alpha k+1)}\int\limits_0^{\beta(\alpha,k)\delta^{\alpha k+2}}f^*(s')\mathrm{d}s'\mathrm{d}\delta.$$

Finally, the change of variable $s'=\beta(\alpha,k)\rho^{\alpha k+2}$ gives

$$u^\star(r)\leq\int\limits_r^{r_0}\delta^{-(\alpha k+1)}\int\limits_0^{\delta}f^\star(\rho)\rho^{\alpha k+1}\mathrm{d}\rho\mathrm{d}\delta=v(r). \tag{4.44}$$

This is statement (4.3) of Theorem 4.1.

Now, we prove (4.4). Let $\varepsilon>0$. Then, by Hölder's inequality, we get

$$\frac{1}{\varepsilon}\int\limits_{t<|u|\leq t+\varepsilon}|\nabla u|^q\mathrm{d}\mu\leq\left(\frac{1}{\varepsilon}\int\limits_{t<|u|\leq t+\varepsilon}|\nabla u|^2\mathrm{d}\mu\right)^{\frac{q}{2}}\left(\frac{1}{\varepsilon}\int\limits_{t<|u|\leq t+\varepsilon}\mathrm{d}\mu\right)^{\frac{2-q}{2}}.$$

Letting $\varepsilon \to 0$ and using (4.35), we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{|u|>t}|\nabla u|^q \mathrm{d}\mu \leq \left(\int\limits_{|u|>t}|f|\mathrm{d}\mu\right)^{\frac{q}{2}}(-m_u'(t))^{\frac{2-q}{2}}. \tag{4.45}$$

By Hardy–Littlewood and (4.41), we have

$$\begin{aligned}-\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{|u|>t}|\nabla u|^q \mathrm{d}\mu &\leq \left(\int\limits_0^{m_u(t)} f^*(s)\mathrm{d}s\right)^{\frac{q}{2}}(-m_u'(t))^{\frac{2-q}{2}}\\ &\leq C(\alpha,k)^{-q}\,(-m_u'(t))\,(m_u(t))^{\frac{-q(\alpha k+1)}{\alpha k+2}}\left(\int\limits_0^{m_u(t)} f^*(s)\mathrm{d}s\right)^{q}.\end{aligned}$$

Integrating the last inequality between 0 and ∞, we have

$$\begin{aligned}\int\limits_D|\nabla u|^q\mathrm{d}\mu &= \int\limits_0^\infty\left(-\frac{d}{dt}\int\limits_{|u|>t}|\nabla u|^q\mathrm{d}\mu\right)\mathrm{d}t\\ &\leq C(\alpha,k)^{-q}\int\limits_0^\infty(-m_u'(t))\,(m_u(t))^{\frac{-q(\alpha k+1)}{\alpha k+2}}\\ &\quad\times\left(\int\limits_0^{m_u(t)} f^*(s)\mathrm{d}s\right)^{q}\mathrm{d}t\\ &= C(\alpha,k)^{-q}\int\limits_0^{\mu(D)}\xi^{\frac{-q(\alpha k+1)}{\alpha k+2}}\left(\int\limits_0^{\xi} f^*(s)\mathrm{d}s\right)^{q}\mathrm{d}\xi\\ &= (\alpha k+2)\beta(\alpha,k)\int\limits_0^{r_0} r^{(1-q)(\alpha k+1)}\left(\int\limits_0^{r} f^\star(\rho)\rho^{\alpha k+1}\mathrm{d}\rho\right)^{q}\mathrm{d}r\\ &= \int\limits_{S_0}|\nabla v|^q\mathrm{d}\mu\end{aligned}$$

which is the desired result (4.4).

4.3.2. Proof of Theorem 4.2. By Theorem 4.1 we have

$$\operatorname{ess\,sup}|u| = u^\star(0)\leq v(0) = C(\alpha,k)^{-2}\int\limits_0^{\mu(D)}\xi^{\frac{-2(\alpha k+1)}{\alpha k+2}}\int\limits_0^{\xi} f^*(s')\mathrm{d}s'\mathrm{d}\xi.$$

Using Hölder's inequality, we get

$$\text{ess sup}\,|u| \leq C(\alpha,k)^{-2} \int_0^{\mu(D)} \xi^{\frac{-\alpha k}{\alpha k+2}-\frac{1}{p}}\mathrm{d}\xi \left(\int_0^{\mu(D)} (f^*(s))^p \,\mathrm{d}s \right)^{\frac{1}{p}} \tag{4.46}$$

which is ineq. (1) of our theorem.

From ineq. (4.4) of Theorem 4.1, we have

$$\int_D |\nabla u|^q \mathrm{d}\mu \leq C(\alpha,k)^{-q} \int_0^{\mu(D)} \left(\frac{1}{\xi} \int_0^{\xi} f^*(s)\mathrm{d}s \right)^q \xi^{\frac{q}{\alpha k+2}}\mathrm{d}\xi. \tag{4.47}$$

Finally, using the Bliss inequality (see ineq. (3) in [9] or (23a) in [51]) we get (2) of our theorem.

5. The Saint-Venant Principle for Relative Torsional Rigidity and other Inequalities

In this part we are interested in the mathematical quantity given by

$$P_\alpha = \int_D vh\mathrm{d}A,$$

where v is a solution of the Dirichlet boundary value problem

$$\mathcal{P}_5 : \begin{cases} -\Delta v = h & \text{in } D \\ v = 0 & \text{on } \partial D. \end{cases}$$

Now if we let $v = h\,w$, with w being a function in $C^2(D)$ satisfying the boundary condition $w = 0$ on $\partial D \cap \mathcal{W}$, then a short computation leads to w being a solution of the problem

$$\mathcal{P}_6 : \begin{cases} -div(h^2 \nabla w) = h^2 & \text{in } D \\ w = 0 & \text{on } \partial D \cap \mathcal{W}. \end{cases}$$

This is exactly the problem $\mathcal{P}_3$ of Sect. 4, with $k = 2$ and $f \equiv 1$. With this substitution, it is clear that

$$P_\alpha = \int_D w\,\mathrm{d}\mu,$$

where the measure $\mathrm{d}\mu = h^2\mathrm{d}A$ is as defined in Sect. 4. In this form P_α can be interpreted as torsional rigidity in dimension $(2\alpha + 2)$, as already expounded on in Sect. 2 for the particular cases $\alpha = 1$ and $\alpha = 2$. We call P_α the relative torsional rigidity of D.

For $\phi \in W_0^{1,2}(D)$, we have

$$\begin{aligned}\int_D \phi\, h \,\mathrm{d}A &= \int_D \phi\,(-\Delta v)\,\mathrm{d}A\\ &= \int_D \nabla\phi\cdot\nabla v\,\mathrm{d}A. \end{aligned}\tag{5.1}$$

Applying the Cauchy–Schwarz inequality we get

$$\begin{aligned}\left(\int_D \phi\, h\,\mathrm{d}A\right)^2 &\leq \left(\int_D |\nabla\phi|\,|\nabla v|\,\mathrm{d}A\right)^2\\ &\leq \int_D |\nabla\phi|^2\,\mathrm{d}A \int_D |\nabla v|^2\,\mathrm{d}A. \end{aligned}\tag{5.2}$$

Since

$$P_\alpha = \int_D v\, h\,\mathrm{d}A = \int_D |\nabla v|^2 \mathrm{d}A, \tag{5.3}$$

we obtain,

$$\frac{\left(\int_D \phi h\,\mathrm{d}A\right)^2}{\int_D |\nabla\phi|^2\,\mathrm{d}A} \leq P_\alpha. \tag{5.4}$$

In fact, by the following theorem, one can define P_α via the variational formulation

$$\frac{1}{P_\alpha} = \inf\left\{F_\alpha(\phi) = \frac{\int_D |\nabla\phi|^2\,\mathrm{d}A}{\left(\int_D \phi h\,\mathrm{d}A\right)^2}:\ \phi\in W_0^{1,2}(D), \phi\not\equiv 0\right\}. \tag{5.5}$$

Theorem 5.1. *Let D be a bounded domain, completely contained in the sector $\mathcal{W}$, and $\phi \in W_0^{1,2}(D)$. Then ϕ is a critical point of the functional*

$$F_\alpha(\phi) = \frac{\int_D |\nabla\phi|^2\,\mathrm{d}A}{\left(\int_D \phi h\,\mathrm{d}A\right)^2}, \tag{5.6}$$

if and only if there exist a constant c such that $\phi = c\,v$, where v solves $\mathcal{P}_5$.

Proof. We first note that

$$F_\alpha(\gamma\phi) = F_\alpha(\phi),$$

for all $\gamma \neq 0$. Suppose ϕ is a critical point of F_α. One can then reduce the problem of minimizing the functional F_α to the problem of finding functions which realize the infimum of $\int_D |\nabla\phi|^2 \mathrm{d}A$ under the constraint $\int_D \phi h \mathrm{d}A = 1$ (cf. [12]). Critical points then satisfy

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \int_D |\nabla(\phi+\epsilon\psi)|^2 \mathrm{d}A = c\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \int_D (\phi+\epsilon\psi)\, h \mathrm{d}A, \tag{5.7}$$

where c is a Lagrange multiplier, for all $\psi \in W_0^{1,2}(D)$. Since $C_0^\infty(D)$ is dense in $W_0^{1,2}(D)$ one needs only to treat the case $\phi, \psi \in C_0^\infty(D)$. Simplifying (5.7) one obtains

$$\int_D \langle \nabla\phi, \nabla\psi \rangle \mathrm{d}A = \frac{c}{2} \int_D \psi h \mathrm{d}A. \tag{5.8}$$

Using Green's formula gives

$$\int_D \langle \nabla\phi, \nabla\psi \rangle \mathrm{d}A = \int_D (-\triangle\phi)\psi \,\mathrm{d}A. \tag{5.9}$$

Combining (5.8) and (5.9) leads to

$$\int_D \left(\triangle\phi + \frac{c}{2} h\right) \psi \mathrm{d}A = 0,$$

for all $\psi \in C_0^\infty(D)$. Therefore,

$$\triangle\phi + \frac{c}{2} h = 0 \tag{5.10}$$

Then, $\frac{2}{c}\phi$ is a solution of $\mathcal{P}_5$. Since the solution of problem $\mathcal{P}_5$ is unique (see Sect. 4), it obtains that

$$\phi = \frac{c}{2} v.$$

The weak solution of the variational problem is indeed a strong solution, as defined by $\mathcal{P}_6$ (or equivalently via $\mathcal{P}_5$). $\square$

Remark 5.2. Using the notation of Sect. 4, this variational formulation is equivalent to

$$\frac{1}{P_\alpha} = \inf_{\phi \in W_2(D, \mathrm{d}\mu)} \frac{\int_D |\nabla\phi|^2 \,\mathrm{d}\mu}{\left(\int_D \phi \,\mathrm{d}\mu\right)^2}, \tag{5.11}$$

where $\mathrm{d}\mu = h^2 \,\mathrm{d}A$.

To see this, we let $v = h\, w$ as before. Then w solves $\mathcal{P}_6$. We note as before that (compare with (3.30))

$$\begin{aligned} \int_D |\nabla v|^2 \,\mathrm{d}A &= \int_D |\nabla w|^2 \mathrm{d}\mu \\ &= \int_D \langle h^2 \nabla w, \nabla w \rangle \mathrm{d}A \\ &= -\int_D div\left(h^2 \nabla w\right) \, w \mathrm{d}A \\ &= \int_D h^2 \, w \mathrm{d}A \end{aligned}$$

$$= \int_D w \mathrm{d}\mu$$
$$= \int_D v\, h\, \mathrm{d}A. \tag{5.12}$$

Whence,

$$\frac{1}{P_\alpha} = \frac{\int_D |\nabla w|^2 \,\mathrm{d}\mu}{\left(\int_D w \,\mathrm{d}\mu\right)^2}. \tag{5.13}$$

Finally, the same computation used to prove (5.4) gives, for all $\phi \in W_2(D, \mathrm{d}\mu)$,

$$\frac{\left(\int_D \phi \,\mathrm{d}\mu\right)^2}{\int_D |\nabla \phi|^2 \,\mathrm{d}\mu} \le P_\alpha. \tag{5.14}$$

We now prove new results for relative torsional rigidity without recourse to the dimensional interpretation in Weinstein fractional space of Sect. 2.

Theorem 5.3. *Let D be a bounded domain, with a piecewise smooth boundary, completely contained in $\mathcal{W}$; then*

$$P_\alpha \lambda < A_\alpha, \tag{5.15}$$

where

$$A_\alpha = \int_D h^2 \mathrm{d}A.$$

Proof. The result follows immediately using the Cauchy–Schwarz inequality in the statement

$$P_\alpha = \frac{\left(\int_D v\, h\, \mathrm{d}A\right)^2}{\int_D |\nabla v|^2 \,\mathrm{d}A} \le \frac{\int_D v^2 \mathrm{d}A \int_D h^2 \mathrm{d}A}{\int_D |\nabla v|^2 \,\mathrm{d}A}$$
$$\le \lambda^{-1} A_\alpha.$$

The last inequality was obtained applying the Rayleigh–Ritz principle for λ with v being a test function. □

Remark 5.4. One can emulate the work of Pólya–Szegő (cf. [38], p. 82) to prove this theorem by first expressing P_α and A_α in terms of eigenfunction expansions emanating from the membrane problem. Since the eigenfunctions $\{u_n\}_{n=1}^\infty$ form an orthonormal basis of $L^2(D)$, corresponding to the eigenvalues

$$0 < \lambda \equiv \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \to \infty,$$

one can write

$$A_\alpha = \int_D h^2 \,\mathrm{d}A = \sum_{n=1}^\infty \left(\int_D h\, u_n \,\mathrm{d}A\right)^2, \tag{5.16}$$

and

$$P_\alpha = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \left(\int_D h\, u_n \,\mathrm{d}A \right)^2 . \tag{5.17}$$

The result (5.15) is then immediate from the ordering of the eigenvalues, viz.

$$P_\alpha < \frac{1}{\lambda_1} \sum_{n=1}^{\infty} \left(\int_D h\, u_n \,\mathrm{d}A \right)^2 = \frac{1}{\lambda_1} A_\alpha .$$

To obtain (5.16), first expand $h = \sum_{n=1}^{\infty} \alpha_n\, u_n$ with $\alpha_n = \int_D h u_n \,\mathrm{d}A$ and then use Plancherel–Parseval. To obtain (5.17), we expand the function $v = \sum_{n=1}^{\infty} \beta_n\, u_n$. Inserting into $\mathcal{P}_5$ leads to $\beta_n = \frac{1}{\lambda_n} \int_D h\, u_n \,\mathrm{d}A$. The statement (5.17) is immediate by virtue of (5.3).

Theorem 5.5. *Let $D \subset \mathcal{W}$ be a bounded domain, with a piecewise smooth boundary; then*

$$P_\alpha \lambda^{\alpha+2} \geq \frac{\pi}{\alpha} j_{\alpha,1}^{2\alpha}. \tag{5.18}$$

Remark 5.6. The proof of this theorem is reminiscent of the work of Payne and Rayner [35].

Proof. As before, let u be the first eigenfunction of the Dirichlet problem. By the variational formulation one can see that

$$\frac{1}{P_\alpha} \leq \frac{\int_D |\nabla u|^2 \,\mathrm{d}A}{\left(\int_D uh \,\mathrm{d}A\right)^2} \tag{5.19}$$

Using (3.2) we obtain

$$\frac{1}{P_\alpha} \leq \frac{2\alpha}{\pi} \lambda^{\alpha+1} \frac{\int_0^{j_{\alpha,1}} r J_\alpha^2(r) \mathrm{d}r}{\left(\int_0^{j_{\alpha,1}} r^{\alpha+1} J_\alpha(r) \mathrm{d}r\right)^2} \frac{\int_D |\nabla u|^2 \,\mathrm{d}A}{\int_D u^2 \,\mathrm{d}A} \tag{5.20}$$

$$= \frac{2\alpha}{\pi} \lambda^{\alpha+2} \frac{j_{\alpha,1}^2 J_{\alpha+1}^2(j_{\alpha,1})}{2 j_{\alpha,1}^{2\alpha+2} J_{\alpha+1}^2(j_{\alpha,1})} \tag{5.21}$$

$$= \frac{\alpha}{\pi j_{\alpha,1}^{2\alpha}} \lambda^{\alpha+2}. \tag{5.22}$$

□

The next result is the Saint-Venant principle for wedge-like membranes.

Theorem 5.7. *Let D be bounded domain completely contained in $\mathcal{W}$, with a piecewise smooth boundary; then*

$$P_\alpha \leq \frac{1}{\alpha+2} \left(\frac{\alpha}{(4\alpha+4)^\alpha \pi} \left(\int_D h^2 \mathrm{d}A \right)^{\alpha+2} \right)^{\frac{1}{\alpha+1}} . \tag{5.23}$$

Equality is attained for the perfect sector.

Proof. Let v be the solution of problem $\mathcal{P}_5$. As before, using the substitution $v = h\,w$, w is then the unique solution of problem $\mathcal{P}_6$. We let $w_\star$ be the solution of the symmetrized problem for $\mathcal{P}_6$ for $k = 2$ on the symmetrized domain S_0 (as detailed in Sect. 4). Then, by Theorem 4.1, one obtains

$$\int_D |\nabla w|^2 \mathrm{d}\mu \leq \int_{S_0} |\nabla w_\star|^2 \mathrm{d}\mu. \tag{5.24}$$

A little computation gives $v_\star = w_\star h$, the unique solution of the problem

$$\begin{cases} -\Delta v = h & \text{in } S_0 \\ v = 0 & \text{on } \partial S_0. \end{cases}$$

Let r_0 be the radius of the sector S_0 as before. It is not difficult to check that

$$v_\star(r,\theta) = \frac{1}{4\alpha + 4}(r_0^2 - r^2)\, h(r,\theta) \quad \forall (r,\theta) \in S_0. \tag{5.25}$$

Using the definition of P_α and the fact that

$$\int_{S_0} |\nabla w_\star|^2 \mathrm{d}\mu = \int_{S_0} |\nabla v_\star|^2 \mathrm{d}A = \int_{S_0} v_\star h \mathrm{d}A \tag{5.26}$$

we obtain

$$P_\alpha \leq \int_{S_0} v_\star h \mathrm{d}A \tag{5.27}$$

$$= \frac{\pi r_0^{2\alpha+4}}{16\alpha(\alpha+1)^2(\alpha+2)}. \tag{5.28}$$

Finally, combining this with

$$r_0 = \left[\frac{4\alpha(\alpha+1)}{\pi} \int_D h^2 \mathrm{d}A\right]^{\frac{1}{2\alpha+2}}, \tag{5.29}$$

we obtain our result (5.23). Equality for the case of the sector follows from considerations in Sect. 4. □

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