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A sharp upper bound for the first Dirichlet eigenvalue of a class of wedge-like domains

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Abstract. By introducing new geometric factors which lend themselves to the Payne interpretation in Weinstein fractional space, we prove new isoperimetric inequalities which complement those of Payne–Weinberger and Saint-Venant giving a new upper bound for the fundamental mode of vibration of a wedge-like membrane and a new lower bound for its "relative torsional rigidity". We also prove a new weighted version of a result of Crooke–Sperb for the associated fundamental eigenfunction of the Dirichlet Laplacian for such domains. A new weighted Rellich-type identity for wedge-like domains is also proved to achieve this latter task.

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1. Introduction

Throughout this article, D will denote a planar domain given in polar coordinates by

$$D = \left\{ (r, \theta) \,|\, 0 < r < \rho(\theta), \;\; 0 < \theta < \frac{\pi}{\alpha} \right\}. \tag{1.1}$$

Then D is a bounded domain completely contained in the wedge $\mathcal{W}$ of angle $\frac{\pi}{\alpha}$, $\alpha \geq 1$. We will denote by Γ the curved part of its boundary defined by $r = \rho(\theta)$, $\;0 \leq \theta < \frac{\pi}{\alpha}$. Let u be the fundamental eigenfunction of the Dirichlet problem in D, and λ its associated fundamental eigenvalue:

$$\Delta u + \lambda u = 0 \text{ in } D, \quad u = 0 \text{ on } \Gamma. \tag{1.2}$$

When Γ is convex, we know that $(x, n) > 0$, [1,12,17,24,27,33] where n is the unit outward normal vector to Γ at x a radius vector on Γ, where $(\cdot, \cdot)$ denotes the usual dot product. For such domains, we define

$$B_\alpha = \int_\Gamma \frac{1}{(x, n)} h^2 \mathrm{d}\sigma, \qquad A_\alpha = \int_D h^2 \mathrm{d}A \tag{1.3}$$

Here, we let $\|x\| = r = \rho(\theta)$ and $h = r^\alpha \sin \alpha\theta$. We note that h is a harmonic nonnegative function in $\mathcal{W}$. By a perfect sector [30], we mean the circular sector

$$S_0 = \left\{ (r, \theta) \;\middle|\; 0 < r < R_0, \; 0 < \theta < \frac{\pi}{\alpha} \right\}$$

with R_0 the radius such that $|S_0| = \int_{S_0} h^2 \mathrm{d}A = A_\alpha$.

The introduction of these geometric quantities stems from the desire to prove counterparts to the weighted Faber–Krahn (or Payne and Weinberger [30]) and weighted Saint-Venant inequalities [20] when the domain D is restricted to a wedge.

The purpose of this paper is to prove the following theorem.

Birkhäuser

Theorem 1.1 (Main Theorem). *The fundamental eigenvalue* λ *of the wedge-like membrane* D *defined by* (1.1) *satisfies the inequality*

$$\lambda \le \frac{B_\alpha}{(2\alpha+2)A_\alpha} j_{\alpha,1}^2. \tag{1.4}$$

Equality holds if and only if D *is a perfect sector* S_0.

We provide two examples. In the first, we treat a right triangle, with angles $\frac{\pi}{2}, \frac{\pi}{\alpha}, \frac{(\alpha-2)\pi}{2\alpha}$ and unit hypotenuse (when $\alpha > 2$), and discuss our bounds in comparison with all existing ones. In the second, we provide explicit upper and lower bounds for the fundamental eigenvalue of a regular α-polygon with n sides, for $2 \le n \le 15$, $\alpha = 1, 1.5, 2, 3, 4, 5,$ and 6. This is a polygon with $(n+2)$ sides sitting on a slice of the unit circular sector of opening π/α, with n equal sides joining points on the circular sector, and the first and last edges closing at the origin (when $\alpha = 1$, and this is a regular semipolygon with n sides inscribed in the unit semicircle).

A bit of background is necessary before we state our contribution. The domain $D \subset \mathbb{R}^2$ is said to be strictly star shaped (with respect to a point inside D) when $(\xi, n) > 0$ for all $\xi \in \partial D$. For such domains, Pólya and Szegő [33] introduced the geometric factor B

$$B = \int_{\partial D} \frac{1}{(\xi, n)} \, \mathrm{d}\xi.$$

Here ξ denotes a point on the boundary ∂D, n is the outward normal at ξ, and (ξ, n) is the "support function" of ∂D. This geometric factor has been extensively studied in the literature; see [1] and the more recent works of Freitas and Krejčiřik [17] and Laugesen and Siudeja [27]. For such domains, Pólya and Szegő [33] proved

$$\lambda \le j_{0,1}^2 \frac{B}{2|D|}, \tag{1.5}$$

($|D|$ denotes the area of D.) Equality holds if and only if D is a disk centered at the origin. For torsional rigidity, P, they also proved

$$P \ge \frac{B}{|D|^2} \tag{1.6}$$

with equality holding for all ellipses. Here we use the standard notation for the Bessel function and its zeros [2]. Inequality (1.5) has been recently extended, for a star-shaped domain $D \subset \mathbb{R}^d$ in Freitas and Krejčiřik [17], to

$$\lambda \le j_{d/2-1,1}^2 \frac{B}{d\,|D|}. \tag{1.7}$$

They also generalized (1.6) to the d-dimensional setting in [17]. In the same vein, Crooke and Sperb [12] (see also [24]) proved for a star-shaped $D \subset \mathbb{R}^2$

$$\frac{\|u\|_2^2}{\|u\|_1^2} \ge \frac{\lambda}{2B} \tag{1.8}$$

using the Rellich identity [36] (see also [23]),

$$\int_{\partial D} (\xi, n) \left(\frac{\partial u}{\partial n}\right)^2 \mathrm{d}\xi = 2\lambda \int_D u^2 \mathrm{d}x. \tag{1.9}$$

Herein, we set $\|u\|_p = \left(\int_D u^p \mathrm{d}x\right)^{1/p}$, for $p > 0$. Ineq. (1.8) is a counterpart to the $p = 1$, $q = 2$ case of the Chiti reverse Hölder inequality [10,11] first proved by Payne and Rayner [29]

$$\frac{\|u\|_q}{\|u\|_p} \le K(p,q,d)\, \lambda^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \text{ for } q \ge p > 0. \tag{1.10}$$

Here

$$K(p,q,d) = (d\,C_d)^{\frac{1}{q}-\frac{1}{p}}\, j_{\frac{d}{2}-1,1}^{d\,(\frac{1}{q}-\frac{1}{p})}\, \frac{\left(\int_0^1 r^{d-1+q(1-\frac{d}{2})} J_{\frac{d}{2}-1}^q(j_{\frac{d}{2}-1,1}\, r)\mathrm{d}r\right)^{\frac{1}{q}}}{\left(\int_0^1 r^{d-1+p(1-\frac{d}{2})} J_{\frac{d}{2}-1}^p(j_{\frac{d}{2}-1,1}\, r)\mathrm{d}r\right)^{\frac{1}{p}}}.$$

The geometric factor B is central to a collection of new isoperimetric inequalities recently proved, for star-shaped domains in $\mathbb{R}^d$, by Laugesen and Siudeja [27] in the spirit of the Freitas and Krejčiřik [17] inequality above. Laugesen and Siudeja [27] proved, for instance, that $(\sum_{i=1}^n \lambda_i)|D|^{2/d}/B$, which is scale invariant, is maximized by the ball in $\mathbb{R}^d$, for $n \geq 1$, when D is star-shaped. In fact, this is also the case of root-sums of the eigenvalues of the Dirichlet problem, geometric means of the first eigenvalues, and convex trace sums of the eigenvalues, when scaled by $|D|^{2/d}/B$ are also maximized by the ball in this case as well.

Ineq. (1.7) is clearly the counterpart to the Faber–Krahn inequality [14,26]

$$\lambda \geq \frac{C_d^{2/d}\, j_{d/2-1,1}^2}{|D|^{2/d}} \tag{1.11}$$

where $j_{d/2-1,1}$ denotes the first positive zero of the Bessel function $J_{d/2-1}(x)$ and $C_d = \pi^{d/2}/\Gamma(1+d/2)$ is the volume of the unit ball. For a wedge-like domain D, Payne and Weinberger [30] proved

$$\lambda \geq \left(\frac{4\alpha(\alpha+1)}{\pi} A_\alpha\right)^{-1/(\alpha+1)} j_{\alpha,1}^2. \tag{1.12}$$

Most of these results are reviewed in the paper of Ashbaugh and Benguria [4]. As proved by Payne in [28], (1.12) has the interpretation of being a version of the Faber–Krahn inequality (1.11) for domains with axial or biaxial symmetry in Weinstein fractional space of dimension $d = 2\alpha + 2$ [40]. We succeeded in [20] to prove a weighted version of the Saint-Venant inequality for relative torsional rigidity, P_α, which has the same interpretation as the Saint-Venant inequality for regular torsional rigidity for these domains with axial or biaxial symmetry in Weinstein fractional space of dimension $d = 2\alpha + 2$. It states that

$$P_\alpha \leq \frac{1}{\alpha+2} \left(\frac{\alpha}{(4\alpha+4)^\alpha \pi}\right)^{\frac{1}{\alpha+1}} A_\alpha^{\frac{\alpha+2}{\alpha+1}}, \tag{1.13}$$

with equality attained for the perfect sector.

We also mention the related work of Sperb [39]. By tweaking the definitions of the geometric factors B_α and A_α defined in [39], we obtain new results which have the right interpretation of being versions of the Freitas and Krejčiřik [17] inequality (1.7) and Crooke and Sperb [12] inequality (1.8) in Weinstein fractional space of dimension $d = 2\alpha + 2$. While relative torsional rigidity lends itself to interpretation in Weinstein fractional space, torsional rigidity does not. This is due to the fact that the definition of former involves the harmonic function h, while the latter does not. We introduced the concept of relative torsional rigidity in [20] for a problem first discussed by Philippin [31]. The introduction of relative torsional rigidity helps improve the Payne–Weinberger inequality, a problem we plan to come back to in a future work [21]. In fact for wedge-like domains, relative torsional rigidity has the interpretation of being regular torsional rigidity for axi-symmetric domains, as in the discussion in Sect. 2 below, a matter which we expounded on in [20].

We note that Ratzkin has proved a version of the Payne–Weinberger inequality in higher dimensions in [35] in the context of convex cones. Raghoub proved in his Ph.D. Thesis [34] a version of this Payne–Weinberger inequality for subdomains of the upper half-space of $\mathbb{R}^3$ and orthants (also in $\mathbb{R}^3$). Brock, Chiacchio, and Mercaldo pushed this result in higher dimensions as well, and their offer offers a less general alternative to [35]. We have also succeeded in generalizing the inequalities presented in this paper in higher dimensions in our upcoming article [22].

Now we focus on the interpretation in the cases $\alpha = 1, 2$. We follow the tradition of Weinstein and Payne by invoking transformations rather than interpret the degenerate operators in cylindrical coordinates. This is a standard trick detailed in the classical book of Gilbert [19]. Throughout this paper, we assume to be a smooth positive function.

Our new inequalities, stated and proved in the following, are counterparts to the Payne–Weinberger inequality (1.12), weighted Saint-Venant inequality (1.13), and the Chiti reverse Hölder inequalities for a wedge-like membrane described by (1.1). In the case of the Chiti reverse Hölder inequalities, we focus on the interpretation of Payne in Weinstein fractional space rather than the full proof in all cases of $\alpha > 1$. The proof without reference to the interpretation is readily available in our article [20] (see also the discussion below).

The counterpart of (1.12) follows the spirit of Freitas and Krejčiřik (1.7), while the new relative rigidity inequality follows the spirit of (1.6).

After motivating these new results via the Payne interpretation in Weinstein space in Sect. 2, we derive basic isoperimetric inequalities satisfied by B_α in Sect. 3. We next prove the upper bound estimate for the fundamental mode of vibration λ of the wedge-like domain D satisfying (1.1) in Sect. 4. We then prove the lower bound estimate for relative torsional rigidity P_α in Sect. 5. A new weighted Rellich-type identity is proved in Sect. 6 and is used to produce a new weighted Crooke–Sperb inequality, independently of the interpretation in Weinstein fractional space considerations.

2. Payne interpretation in Weinstein fractional space

Following Payne, who first introduced the Weinstein method [28,40] of interpretation, we will motivate a lot of our work by showing concretely how to obtain, via appropriate interpretation of the various inequalities in the correct higher-dimensional space, various isoperimetric inequalities when our wedge consists of half-space and quarter-space. We will specifically focus on the following inequalities: Chiti [10,11], Freitas and Krejčiřik [17], and Crooke and Sperb [12]. These are, respectively, inequalities (1.10), (1.7), and (1.8). This interpretation in higher Weinstein fractional space was pioneered by Payne [28] who used it in the case of the Faber–Krahn [14,26] inequality to motivate and discover via the right interpretation in a higher-dimensional setting results that later became the Payne–Weinberger inequality for a wedge-like membrane [30]. We exploited it in [20] to discover and motivate similar results for torsional rigidity fine tuning the Saint-Venant inequality [13,32].

We note that the actual full proof does not depend on this interpretation and is independently carried out using a weighted symmetrization procedure in combination with the co-area formula exploiting a weighted isoperimetric inequality in [30] for the case of weighted Faber–Krahn inequality (also called the Payne–Weinberger inequality (see also [28]). This very program, independently of the Payne interpretation, has been carried out by the authors in [20] in the case of the weighted Chiti inequality where we proved, also for D a bounded domain contained in the wedge $\mathcal{W}$, and for p, q be real numbers such that $q \geq p > 0$, that

$$\left(\int_D u^q h^{2-q} \mathrm{d}A \right)^{\frac{1}{q}} \leq K(p,q,\lambda,\alpha) \left(\int_D u^p \, h^{2-p} \mathrm{d}A \right)^{\frac{1}{p}} \tag{2.1}$$

with

$$K(p,q,\lambda,\alpha) = \left(\frac{\pi}{2\alpha}\right)^{\frac{p-q}{pq}} \lambda^{(\alpha+1)\frac{q-p}{pq}} \frac{\left(\int_0^{j_{\alpha,1}} r^{(2-q)\alpha+1} J_\alpha^q(r)\mathrm{d}r\right)^{\frac{1}{q}}}{\left(\int_0^{j_{\alpha,1}} r^{(2-p)\alpha+1} J_\alpha^p(r)\mathrm{d}r\right)^{\frac{1}{p}}}.$$

The case $q = 2$, $p = 1$, particularly pertinent for this paper, and providing an explicit form for the constant $K(1, 2, \lambda, \alpha)$, reads

$$\int_D u^2 \mathrm{d}A \leq \frac{\alpha}{\pi j_{\alpha,1}^{2\alpha}} \lambda^{\alpha+2} \left(\int_D u\, h \mathrm{d}A \right)^2 . \tag{2.2}$$

For domains in higher dimensions, this Payne interpretation has been exploited by Raghoub in his 1996 Ph.D. Thesis [34] to prove Faber–Krahn-type inequalities for D a subdomain of the upper half-space of $\mathbb{R}^3$ and orthants (also in $\mathbb{R}^3$); see also [25] where some of Ragoub's results were announced. We also note recent activity by Brock et al. [6–8], and the work of Ratzkin [35] which unifies many of these ideas in the context of convex cones in higher dimensions, independently of the Payne interpretation. The authors push this circle of ideas, also for convex cones in higher dimensions, in their upcoming article [22]. We also point out the paper [5] for similar consideration in higher dimensions.

We now focus on the interpretation in the cases $\alpha = 1, 2$. We follow the tradition of Weinstein and Payne by invoking transformations rather than interpret the degenerate operators in cylindrical coordinates. This is a standard trick detailed in the classical book of Gilbert [19]. We also assume ρ to be a smooth positive function.

Let $\mathcal{W}_1 = \{(x, y) \in \mathbb{R}^2, y > 0\}$ denote the upper half-space, and let $\mathcal{W}_2 = \{(x, y) \in \mathbb{R}^2, x > 0, y > 0\}$ denote the quarter-space. We will treat the cases when D is a subset of either. We recall that the Dirichlet eigenvalue problem is given by

$$-\Delta u = \lambda u \text{ in } D, \quad u = 0 \text{ on } \partial D. \tag{2.3}$$

(a) Case of the half-space.

When $D \subset \mathcal{W}_1$, the first eigenfunction can be represented as

$$u = y\, w, \tag{2.4}$$

where w is a positive smooth function vanishing on $\partial D \cap \mathcal{W}_1$. Substituting into (2.3) and simplifying gives

$$\Delta w + \frac{2}{y}\frac{\partial w}{\partial y} = \lambda w \quad \text{ in } D, w = 0 \text{ on } \partial D \cap \mathcal{W}_1. \tag{2.5}$$

Let $\Phi(x_1, x_2, x_3, x_4)$ be the function defined in $\mathbb{R}^4$ by

$$\Phi(x_1, x_2, x_3, x_4) = w(x, y) \quad \text{ where } \quad x = x_1; y = \sqrt{x_2^2 + x_3^2 + x_4^2};$$

This function has axial symmetry with respect to the x_1-axis. It is defined on

$$D_4 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x = x_1, \quad y = \sqrt{x_2^2 + x_3^2 + x_4^2}, \quad (x, y) \in D\} \cup \{O\}.$$

D_4 is obtained from D via rotation around the x_1-axis in $\mathbb{R}^4$ (and O is the origin in $\mathbb{R}^4$). The function Φ satisfies

$$-\Delta_4 \Phi = \lambda\Phi \text{ in } D_4, \quad \Phi = 0 \text{ on } \partial D_4.$$

Indeed

$$\begin{aligned}
\Delta_4 \Phi &= \sum_{i=1}^4 \frac{\partial^2 \Phi}{\partial x_i^2} \\
&= \frac{\partial^2 w}{\partial x^2} + \sum_{i=2}^4 \frac{x_i}{y}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial y}\right) + \sum_{i=2}^4 \frac{y^2 - x_i^2}{y^3}\frac{\partial w}{\partial y} \\
&= \Delta w + \frac{2}{y}\frac{\partial w}{\partial y} \\
&= -\lambda w \\
&= -\lambda \Phi.
\end{aligned}$$

In fact, we can say more.

Lemma 2.1. *λ is the fundamental eigenvalue of the Dirichlet problem in D_4.*

Proof. Since u is positive, so is the case of w and Φ. Since the fundamental eigenvalue is characterized as being the only eigenvalue with eigenfunction of constant sign (Corollary 2 in Chavel [9], p. 20), it transpires that Φ is the fundamental eigenfunction of the Dirichlet problem in D_4, and λ is its corresponding eigenvalue. $\square$

Rather than invoke cylindrical or spherical coordinates to explain the forms of the Laplacian and the transformations, we are performing below, we follow closely the work of Payne [28], Ragoub [34], and Weinstein [40] who cast these in a much more general setting of axially symmetric systems. Let Ψ be the map from $D \times (0, 2\pi) \times (0, \pi) \to D_4$ defined by

$$\Psi(x, y, \theta, \varphi) = (x_1, x_2, x_3, x_4),$$

where

$$\begin{cases} x_1 = x \\ x_2 = y \cos\theta \sin\varphi \\ x_3 = y \sin\theta \sin\varphi \\ x_4 = y \cos\varphi. \end{cases}$$

Ψ is a diffeomorphism, and its Jacobian is given by

$$\mathcal{J}_\Psi(x, y, \theta, \varphi) = -y^2 \sin\varphi < 0.$$

Using standard theorems, we change variables to obtain

$$\int_{D_4} \mathrm{d}V_4 = 4\pi \int_D y^2 \mathrm{d}V.$$

For the case of the Chiti inequality for half-space, we first write the statement in the 4-dimensional domain D_4

$$\left[\int_{D_4} \Phi^q \mathrm{d}V_4\right]^{\frac{1}{q}} \leq K(p, q, 4) \left[\int_{D_4} \Phi^p \mathrm{d}V_4\right]^{\frac{1}{p}}$$

for $q \geq p \geq 0$. As before, changing variables, for $p > 0$, yields

$$\left[\int_{D_4} \Phi^p \mathrm{d}V_4\right]^{\frac{1}{p}} = (4\pi)^{\frac{1}{p}} \left[\int_D w^p y^2 \mathrm{d}V\right]^{\frac{1}{p}}.$$

Using (1.10), we get

$$\left[\int_D w^q y^2 \mathrm{d}V\right]^{\frac{1}{q}} \leq (4\pi)^{\frac{1}{p}-\frac{1}{q}} K(p, q, 4) \left[\int_D w^p y^2 \mathrm{d}V\right]^{\frac{1}{p}},$$

with equality if and only if D is a half-disk.

We next treat the counterpart to the Faber–Krahn inequality, namely the Freitas and Krejčiřik inequality (1.7) for half-space. It is a bit more delicate. Let D be the domain which lies in the half-space $\mathcal{W}_1$ such that its curved boundary Γ is some smooth polar curve: $r = \rho(\omega), \omega \in (0, \pi)$ i.e.,

$$D = \left\{(x, y) \in \mathcal{W}_1 \mid \quad 0 < \sqrt{x^2 + y^2} < \rho\left(\arccos \frac{x}{\sqrt{x^2 + y^2}}\right)\right\}.$$

Lemma 2.2. *D_4 is a strictly star-shaped domain with respect to the origin.*

Remark 2.3. We note that $\lim_{\omega\to 0^+}\rho(\omega)>0$ and $\lim_{\omega\to\pi^-}\rho(\omega)>0$ are implied in the statement of the lemma, and the "singular boundary spikes" $\{(x,0,0,0){:}0<x<\rho(0^+)\}$ and $\{(x,0,0,0){:}-\rho(\pi^-)<x<0\}$ have been removed from the domain D_4. These spikes have capacity zero and hence are removable from D_4 without affecting the eigenvalues. In any case, the proofs we offered in our earlier paper [20] for the weighted Chiti inequality (2.1), and in the main theorems of this paper, are independent and are quite general. They do not require the interpretation in fractional Weinstein space used to motivate them.

Proof. Let $\xi=(x_1,x_2,x_3,x_4)\in D_4$ and $t\in[0,1)$. Then, by the definition of D_4, $(x_1,\sqrt{x_2^2+x_3^2+x_4^2})\in D$, and in light of the definition of D, we obtain that $(tx_1,\sqrt{(tx_2)^2+(tx_3)^2+(tx_4)^2})\in D$. This implies that $t\xi\in D_4$, for $0\le t<1$, and so D_4 is a star-shaped domain, as well.

We next prove that $(\xi,n)>0$, where ξ is position vector on ∂D_4 and n is a unit outward normal to ∂D_4. The boundary of D_4 is of course given by

$$\partial D_4=\left\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid \quad \left(x_1,\sqrt{x_2^2+x_3^2+x_4^2}\right)\in\Gamma\right\}.$$

To begin, consider the map Ψ_1 defined from $(0,\pi)\times(0,2\pi)\times(0,\pi)$ into ∂D_4 by

$$\Psi_1(\omega,\theta,\varphi)=(\rho(\omega)\cos\omega,\rho(\omega)\sin\omega\cos\theta\sin\varphi,\rho(\omega)\sin\omega\sin\theta\sin\varphi,\rho(\omega)\sin\omega\cos\varphi).$$

Then Ψ_1 is a parametrization of the hypersurface ∂D_4. The tangent space of ∂D_4 at $\xi\in\partial D_4$ is

$$T_\xi\partial D_4=vect(\partial_\omega\Psi_1,\partial_\theta\Psi_1,\partial_\varphi\Psi_1).$$

The unit outward normal vector to ∂D_4 at ξ, denoted by n, is the normalized vector of the cross-product $[\partial_\omega\Psi_1,\partial_\theta\Psi_1,\partial_\varphi\Psi_1]$. An explicit calculation gives

$$(\xi,n)=\frac{\rho^2}{\sqrt{\dot\rho^2+\rho^2}}>0,$$

as desired. □

Now, we introduce the two mathematical quantities

$$B(D_4)=\int_{\partial D_4}\frac{1}{(\xi,n)}\mathrm{d}A,\quad B_1=\int_\Gamma\frac{y^2}{(z,\tilde n)}\mathrm{d}\sigma,$$

where ξ is the position vector on ∂D_4, n is the unit outward normal to ∂D_4 at ξ, and $\mathrm{d}A$ is the measure on ∂D_4. The corresponding notation for Γ is as follows: z (position vector), $\tilde n$ (unit outward normal to Γ at z), and $d\sigma$ being the measure. Using the above parametrization for ∂D_4, we get

$$\begin{aligned}B(D_4)&=\int_{\partial D_4}\frac{1}{(\xi,n)}\mathrm{d}A=\int_0^\pi\int_0^{2\pi}\int_0^\pi\frac{\sqrt{\dot\rho^2+\rho^2}}{\rho^2}\mid[\partial_\omega\Psi_1,\partial_\theta\Psi_1,\partial_\varphi\Psi_1]\mid\mathrm{d}\omega\mathrm{d}\theta\mathrm{d}\varphi\\&=4\pi\int_0^\pi\left(1+\frac{\dot\rho^2(\omega)}{\rho^2(\omega)}\right)\rho^2(\omega)\sin^2\omega\,\mathrm{d}\omega\\&=4\pi B_1.\end{aligned}$$

Remark 2.4. We note that one could have equally defined $B(D_4)$ in terms of $\rho(\omega)$ is in the second line of the above. As the calculation above suggests the two definitions are equivalent. See the discussion in [27] [formula (6) and Lemma 10.2]. Our work provides a weighted version of the B-functionals of [27].

Now applying the Freitas and Krejčiřik inequality (1.7) in our 4-dimensional domain D_4, we get

$$\lambda \leq \frac{B(D_4)}{4\,\mathrm{vol}(D_4)}\, j_1^2 = \frac{B_1}{4\,A_1}\, j_1^2,$$

where $A_1 = \int_D y^2\,\mathrm{d}V$.

We next do the Weinstein interpretation for the Crooke–Sperb inequality (1.8) to obtain its corresponding sharp form in half-space. We begin by writing this inequality for the 4-dimensional domain D_4

$$\left(\int_{D_4} w\, dV_4\right)^2 \leq \frac{2B(D_4)}{\lambda} \int_{D_4} w^2\,\mathrm{d}V_4.$$

From the above considerations, this simplifies to

$$\left(\int_{D} uy\, dV\right)^2 \leq \frac{2B_1}{\lambda} \int_{D} u^2\,\mathrm{d}V.$$

(b) Case of the quarter-space.

In this case, D is such that $x > 0$ and $y > 0$. The first eigenfunction can be represented as

$$u(x, y) = 2xy\, w(x, y), \tag{2.6}$$

where w is a positive smooth function vanishing on $\partial D \cap \mathcal{W}_2$. Substituting into (2.3), and simplifying, leads to

$$\Delta w + \frac{2}{x}\frac{\partial w}{\partial x} + \frac{2}{y}\frac{\partial w}{\partial y} = \lambda w \quad \text{ in } D, w = 0 \text{ on } \partial D \cap \mathcal{W}_2. \tag{2.7}$$

Now, we introduce the function Ψ defined in $\mathbb{R}^6$ by

$$\Psi(x_1, x_2, x_3, y_1, y_2, y_3) = w(x, y) \quad \text{ where } \quad x = \sqrt{x_1^2 + x_2^2 + x_3^2}; y = \sqrt{y_1^2 + y_2^2 + y_3^2}.$$

This function Ψ is smooth, positive, and is defined on

$$D_6 = \left\{(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 \middle| \left(\sqrt{x_1^2 + x_2^2 + x_3^2}, \sqrt{y_1^2 + y_2^2 + y_3^2}\right) \in D\right\} \cup \{O\},$$

with O being the origin in $\mathbb{R}^6$. It satisfies

$$-\Delta_6\, \Psi = \lambda \Psi \text{ in } D_6, \quad \Psi = 0 \text{ on } \partial D_6.$$

Indeed

$$\begin{aligned}
\Delta_6\, \Psi &= \sum_{i=1}^{3} \frac{\partial^2 \Psi}{\partial x_i^2} + \sum_{i=1}^{3} \frac{\partial^2 \Psi}{\partial y_i^2} \\
&= \sum_{i=1}^{3} \frac{x_i}{x}\frac{\partial}{\partial x_i}\left(\frac{\partial w}{\partial x}\right) + \sum_{i=1}^{3} \frac{x^2 - x_i^2}{x^3}\frac{\partial w}{\partial y} + \sum_{i=1}^{3} \frac{y_i}{y}\frac{\partial}{\partial y_i}\left(\frac{\partial w}{\partial y}\right) + \sum_{i=1}^{3} \frac{y^2 - y_i^2}{y^3}\frac{\partial w}{\partial y} \\
&= \Delta w + \frac{2}{x}\frac{\partial w}{\partial x} + \frac{2}{y}\frac{\partial w}{\partial y} \\
&= -\lambda w \\
&= -\lambda \Psi.
\end{aligned}$$

We also have the following result.

Lemma 2.5. *λ is the fundamental eigenvalue of the Dirichlet problem in D_6.*

Proof. The proof follows the same arguments as in Lemma 2.1. □

Let Ψ be the map from $D \times (0, 2\pi) \times (0, \pi) \times (0, 2\pi) \times (0, \pi) \to D_6$ defined by

$$\Psi(x, y, \theta_1, \varphi_1, \theta_2, \varphi_2) = (x_1, x_2, x_3, y_1, y_2, y_3),$$

where $\begin{cases} x_1 = x\cos\theta_1 \sin\varphi_1 \\ x_2 = x\sin\theta_1\sin\varphi_1 \\ x_3 = x\cos\varphi_1 \\ y_1 = y\cos\theta_2\sin\varphi_2 \\ y_2 = y\sin\theta_2\sin\varphi_2 \\ y_3 = y\cos\varphi_2. \end{cases}$

Ψ is a diffeomorphism. Its Jacobian is explicitly given by

$$\mathcal{J}_\Psi(x, y, \theta_1, \varphi_1, \theta_2, \varphi_2) = x^2y^2 \sin\varphi_1 \sin\varphi_2 > 0.$$

As before, we introduce the geometric factors in their corresponding spaces

$$B(D_6) = \int\limits_{\partial D_6} \frac{1}{(\xi, n)} \mathrm{d}V_6, \quad B_2 = 4\int\limits_{\Gamma} \frac{1}{(z, \tilde{n})} x^2y^2 \mathrm{d}\sigma, \quad A_2 = 4\int\limits_{D} x^2y^2 \mathrm{d}V,$$

where ξ is the position vector on ∂D_6, n is the unit outward normal to ∂D_6 at ξ, and $\mathrm{d}A$ is the measure on ∂D_6. As before, the corresponding notation for Γ is: z (position vector), $\tilde{n}$ (unit outward normal to Γ at z), and $d\sigma$ being the measure. The boundary of D_6 is given by

$$\partial D_6 = \left\{ (x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6 \middle| \left(\sqrt{x_1^2 + x_2^2 + x_3^2}, \sqrt{y_1^2 + y_2^2 + y_3^2} \right) \in \Gamma \right\}.$$

Changing variables, we obtain

$$\mathrm{vol}(D_6) = \int\limits_{D_6} \mathrm{d}V_6 = (4\pi)^2 \int\limits_{D} x^2y^2 \mathrm{d}V = 4\pi^2 A_2.$$

We are now ready for the Weinstein interpretation. For the Chiti inequality on quarter-space, we write (1.10) for the 6-dimensional domain D_6

$$\left[\int\limits_{D_6} \Phi^q \mathrm{d}V_6 \right]^{\frac{1}{q}} \le K(p, q, 6) \left[\int\limits_{D_6} \Phi^p \mathrm{d}V_6 \right]^{\frac{1}{p}} \tag{2.8}$$

for $q \ge p \ge 0$, then simplify, for $p > 0$, by simple change of variables,

$$\left[\int\limits_{D_6} \Phi^p \mathrm{d}V_6 \right]^{\frac{1}{p}} = (4\pi)^{\frac{2}{p}} \left[\int\limits_{D} w^p x^2y^2 \mathrm{d}V \right]^{\frac{1}{p}}.$$

Using this identity, (2.8) reduces to

$$\left[\int\limits_{D} w^q x^2y^2 \mathrm{d}V \right]^{\frac{1}{q}} \le (4\pi)^{\frac{2}{p} - \frac{2}{q}} K(p, q, 6) \left[\int\limits_{D} w^p x^2y^2 \mathrm{d}V \right]^{\frac{1}{p}},$$

with equality if and only if D is a quarter-disk. Now, we do the same for the Freitas and Krejčiřik inequality (1.7) in quarter-space $\mathcal{W}_1$. In this case, D is assumed to be given by

$$D = \left\{ (x, y) \in \mathcal{W}_1 \mid \quad 0 < \sqrt{x^2 + y^2} < \rho \left(\arccos \frac{x}{\sqrt{x^2 + y^2}} \right) \right\}.$$

The arguments of Lemma 2.2 show that D_6 is strictly star-shaped domain with respect to the origin. Let Ψ_2 be the map defined from $(0, \pi) \times (0, 2\pi) \times (0, \pi) \times (0, 2\pi) \times (0, \pi)$ into ∂D_6 by

$$\Psi_2(\omega, \theta_1, \varphi_2, \theta_3, \varphi_4) = (x_1, x_2, x_3, y_1, y_2, y_3),$$

where

$$\begin{cases} x_1 = \rho(\omega) \cos\omega \cos\theta_1 \sin\varphi_1 \\ x_2 = \rho(\omega) \cos\omega \sin\theta_1 \sin\varphi_1 \\ x_3 = \rho(\omega) \cos\omega \cos\varphi_1 \\ y_1 = \rho(\omega) \sin\omega \cos\theta_2 \sin\varphi_2 \\ y_2 = \rho(\omega) \sin\omega \sin\theta_2 \sin\varphi_2 \\ y_3 = \rho(\omega) \sin\omega \cos\varphi_2. \end{cases}$$

This function Ψ_2 is a parametrization of the hypersurface ∂D_6, and using the fact that the unit outward normal vector to D_6 at ξ is the normalized vector of the cross-product $[\partial_\omega, \partial_{\theta_1}, \partial_{\varphi_1}, \partial_{\theta_2}, \partial_{\varphi_2}]$, we get from the parametrization of ∂D_6

$$\begin{aligned} B(D_6) &= \int_{\partial D_6} \frac{1}{(\xi, n)} \mathrm{d}A \\ &= (4\pi)^2 \int_0^\pi \left(1 + \frac{\dot\rho^2(\omega)}{\rho^2(\omega)}\right) \rho^4(\omega) \sin^2\omega \cos^2\omega \, \mathrm{d}\omega \\ &= 4\pi^2 B_2. \end{aligned}$$

Remark 2.6. As in Remark 2.4, one could have equally defined $B(D_6)$ in terms of $\rho(\omega)$; see [27].

Applying (1.7) to our 6-dimensional domain D_6, we get

$$\begin{aligned} \lambda &\le \frac{B(D_6)}{6 \operatorname{vol}(D_6)} \, j_2^2 \\ &= \frac{B_2}{6 \, A_2} \, j_2^2, \end{aligned}$$

We finalize our discussion by producing a version of the Crooke–Sperb inequality which is optimized for quarter-space. From (1.8) for the 6-dimensional domain D_6, we get

$$\left(\int_{D_6} \Phi \, dV_6 \right)^2 \le \frac{2B(D_6)}{\lambda} \int_{D_6} \Phi^2 \, \mathrm{d}V_6,$$

which simplifies immediately to

$$\left(\int_D uyx \, dV \right)^2 \le \frac{2B_2}{\lambda} \int_D u^2 \, \mathrm{d}V.$$

3. A geometric inequality for a wedge-like membrane complementary to Payne–Weinberger

We prove in this section some basic isoperimetric results satisfied by B_α.

Theorem 3.1. *For the wedge-like membrane D defined by (1.1), we have*

$$B_\alpha \geq \frac{\pi}{2\alpha}\left[\frac{4\alpha(\alpha+1)}{\pi}A_\alpha\right]^{\frac{\alpha}{\alpha+1}}. \tag{3.1}$$

Equality holds if and only if D is a perfect sector of angle $\frac{\pi}{\alpha}$.

Remark 3.2. Our result is new and has the interpretation of being a version of an earlier result of Freitas and Krejčiřik [17] in Weinstein space of dimension $d = 2\alpha + 2$ (see Remark 2, p. 2999 of [17]). Our proof is in fact inspired by this remark. We also note an alternative proof in [27] (see Lemma 2.2 therein).

Proof. Using the Cauchy–Schwarz inequality, we get

$$\left(\int_\Gamma h^2 \mathrm{d}\sigma\right)^2 \leq B_\alpha \int_\Gamma (x,n)h^2 \mathrm{d}\sigma. \tag{3.2}$$

The divergence theorem gives

$$\begin{aligned}\int_\Gamma (x,n)h^2\mathrm{d}\sigma &= \int_D \operatorname{div}(h^2 x)\mathrm{d}A\\ &= \int_D \left(2h(\nabla h, x) + 2h^2\right) dA\\ &= (2\alpha+2)A_\alpha.\end{aligned} \tag{3.3}$$

Combining (3.2), (3.3), and the geometric Payne–Weinberger inequality (this is ineq. (2.2) in [30]), we obtain at once

$$\left(\frac{\pi}{2\alpha}\right)^2\left[\frac{4\alpha(\alpha+1)}{\pi}A_\alpha\right]^{\frac{2\alpha+1}{\alpha+1}} \leq \left(\int_\Gamma h^2\mathrm{d}\sigma\right)^2 \leq (2\alpha+2)A_\alpha B_\alpha. \tag{3.4}$$

It is not difficult to see that we have equality for the perfect sector. Now, assume that we have equality in (3.1), then we must have equality in (3.4). Then, by the geometric Payne–Weinberger result [30], D must be a sector of angle $\frac{\pi}{\alpha}$. □

4. An isoperimetric inequality complementary to Payne–Weinberger: Proof of Theorem 1.1 and discussion

In this section, we prove Theorem 1.1, then discuss its consequence for a right triangle with angles $\frac{\pi}{2}, \frac{\pi}{\alpha}, \frac{(\alpha-2)\pi}{2\alpha}$ and unit hypotenuse (in the case $\alpha > 2$).

Proof of the Main Theorem. The proof is inspired by similar considerations in [33]. According to the Rayleigh–Ritz principle, we have

$$\lambda \leq \frac{\int_D |\nabla f|^2 \mathrm{d}A}{\int_D f^2 \mathrm{d}A}, \tag{4.1}$$

where f is any function in the Sobolev space H_0^1. Now, introduce the function v defined in D by

$$v(r,\theta) = g\left(\frac{r}{\rho(\theta)}\right), \tag{4.2}$$

where g is any function of class C^1 such that $g(1) = 0$. Let the function f given by $f = v\,h$ be a test function for the Rayleigh quotient. Then

$$\int_D f^2 \mathrm{d}A = \int_D v^2 h^2 \mathrm{d}A$$
$$= \int_0^{\frac{\pi}{\alpha}} \int_0^{\rho(\theta)} g^2\left(\frac{r}{\rho(\theta)}\right) r^{2\alpha+1} \sin^2 \alpha\theta \, \mathrm{d}r\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{\alpha}} \left(\int_0^{\rho(\theta)} g^2\left(\frac{r}{\rho(\theta)}\right) r^{2\alpha+1}\mathrm{d}r \right) \sin^2 \alpha\theta \, \mathrm{d}\theta.$$

By the change of variable $s = \frac{r}{\rho(\theta)}$, we have

$$\int_D f^2 \mathrm{d}A = \int_0^{\frac{\pi}{\alpha}} \rho^{2\alpha+2}(\theta) \sin^2 \alpha\theta \, \mathrm{d}\theta \int_0^1 g^2(s) s^{2\alpha+1} \, \mathrm{d}s \tag{4.3}$$

A quick computation gives

$$A_\alpha = \int_0^{\frac{\pi}{\alpha}} \int_0^{\rho(\theta)} r^{2\alpha+1} \sin^2 \alpha\theta \, \mathrm{d}r\mathrm{d}\theta$$
$$= \frac{1}{2\alpha+2} \int_0^{\frac{\pi}{\alpha}} \rho^{2\alpha+2}(\theta) \sin^2 \alpha\theta \, \mathrm{d}\theta \tag{4.4}$$

Substituting into (4.3), we get

$$\int_D f^2 \mathrm{d}A = (2\alpha+2) A_\alpha \int_0^1 g^2(s) s^{2\alpha+1} \, \mathrm{d}s. \tag{4.5}$$

For the Dirichlet integral of f, we find

$$\int_D |\nabla f|^2 \mathrm{d}A = \int_D |\nabla v|^2 h^2 \mathrm{d}A$$
$$= \int_0^{\frac{\pi}{\alpha}} \int_0^{\rho(\theta)} \left(\left(\frac{dv}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dv}{d\theta}\right)^2 \right) r^{2\alpha+1} \sin \alpha^2\theta \, \mathrm{d}r\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{\alpha}} \int_0^{\rho(\theta)} \left(\frac{1}{\rho^2(\theta)} + \frac{\dot{\rho}^2(\theta)}{\rho^4(\theta)} \right) g'^2\left(\frac{r}{\rho(\theta)}\right) r^{2\alpha+1} \sin^2 \alpha\theta \, \mathrm{d}r\mathrm{d}\theta$$
$$= \int_0^{\frac{\pi}{\alpha}} \left(\frac{1}{\rho^2(\theta)} + \frac{\dot{\rho}^2(\theta)}{\rho^4(\theta)} \right) \left(\int_0^{\rho(\theta)} g'^2\left(\frac{r}{\rho(\theta)}\right) r^{2\alpha+1}\mathrm{d}r \right) \sin^2 \alpha\theta \, \mathrm{d}\theta$$

By putting $s = \frac{r}{\rho(\theta)}$, we obtain

$$\int_D |\nabla f|^2 \mathrm{d}A = \int_0^{\frac{\pi}{\alpha}} \left(1 + \frac{\dot{\rho}^2(\theta)}{\rho^2(\theta)}\right) \rho^{2\alpha}(\theta) \sin^2 \alpha\theta \, \mathrm{d}\theta \int_0^1 g'^2(s) s^{2\alpha+1} \, \mathrm{d}s \tag{4.6}$$

By the fact that

$$(x, n) = \frac{\rho^2}{\sqrt{\dot{\rho}^2 + \rho^2}}, \quad \texttt{and} \quad \mathrm{d}\sigma = \sqrt{\dot{\rho}^2 + \rho^2} \, \mathrm{d}\theta, \tag{4.7}$$

we find

$$B_\alpha = \int_0^{\frac{\pi}{\alpha}} \left(1 + \frac{\dot{\rho}^2(\theta)}{\rho^2(\theta)}\right) \rho^{2\alpha}(\theta) \sin^2 \alpha\theta \, \mathrm{d}\theta, \tag{4.8}$$

and therefore

$$\int_D |\nabla f|^2 \mathrm{d}A = B_\alpha \int_0^1 g'^2(s) s^{2\alpha+1} \, \mathrm{d}s. \tag{4.9}$$

Combining (4.1), (4.3), and (4.9), we obtain

$$\lambda \leq \frac{B_\alpha}{(2\alpha + 2) A_\alpha} \, \frac{\int_0^1 g'^2(s) s^{2\alpha+1} \, \mathrm{d}s}{\int_0^1 g^2(s) s^{2\alpha+1} \, \mathrm{d}s}. \tag{4.10}$$

Now, we choose the function

$$g(s) = s^{-\alpha} J_\alpha(j_{\alpha,1} s), \tag{4.11}$$

where J_α is the Bessel function of the first kind of order α and $j_{\alpha,1}$ is its first positive zero. Clearly, $g(1) = 0$. With this choice, the inequality (4.10) yields

$$\lambda \leq \frac{B_\alpha}{(2\alpha + 2) A_\alpha} \, j_{\alpha,1}^2. \tag{4.12}$$

Indeed, let u_1 be the first eigenfunction of the Laplace operator with Dirichlet boundary condition in the sector S^1 of angle $\frac{\pi}{\alpha}$ and radius 1; then

$$u_1(r, \theta) = J_\alpha(j_{\alpha,1} r) \sin \alpha\theta = g(r) r^\alpha \sin \alpha\theta. \tag{4.13}$$

Using this, we obtain

$$\begin{aligned} j_{\alpha,1}^2 = \lambda_1(S^1) &= \frac{\int_{S^1} |\nabla u_1|^2 \mathrm{d}A}{\int_{S^1} u_1^2 \mathrm{d}A} \\ &= \frac{\int_0^{\frac{\pi}{\alpha}} \int_0^1 g'^2(r) r^{2\alpha+1} \sin^2 \alpha\theta \, \mathrm{d}r\mathrm{d}\theta}{\int_0^{\frac{\pi}{\alpha}} \int_0^1 g^2(r) r^{2\alpha+1} \sin^2 \alpha\theta \, \mathrm{d}r\mathrm{d}\theta} \\ &= \frac{\int_0^1 g'^2(s) s^{2\alpha+1} \, \mathrm{d}s}{\int_0^1 g^2(s) s^{2\alpha+1} \, \mathrm{d}s} \end{aligned}$$

Plugging this into (4.10), we obtain our desired inequality. In the case of the perfect sector of radius R, it is not difficult to see that

$$\frac{B_\alpha}{(2\alpha + 2) A_\alpha} = \frac{1}{R^2}, \tag{4.14}$$

which ensures equality in this case. □

Remark 4.1. This theorem and the Payne–Weinberger inequality give the two-sided bounds for the fundamental tone of vibration of a drum trapped in a wedge.

$$\left(\frac{4\alpha(\alpha+1)}{\pi}A_\alpha\right)^{\frac{-1}{\alpha+1}} j_{\alpha,1}^2 \le \lambda \le \frac{B_\alpha}{(2\alpha+2)A_\alpha} j_{\alpha,1}^2. \tag{4.15}$$

It is worth noting a few remarks with respect to the two-sided bound (4.15). These bounds are much *en vogue* in recent considerations by various authors [3,15,18,37,38]. The lower bound in (4.15) is Payne–Weinberger's bound (1.12) above, and certainly not the best to date, and the upper bound is the main result of this paper.

To make the notation uniform, we will denote the lower and upper bounds of Siudeja's [38] Theo. 1.1 [formula (1.1)] S_{low} and S_{high}. The upper bound stemming from Pólya and Szegő's work [33] [see formula (3), p. 20] or (1.5) above (PS). The lower and upper bounds coming from Theo. 2 of Freitas' paper [15] are denoted F_{low} and F_{high}. F_{low} is essentially the bound of Payne and Weinberger [30] labeled (1.12) above, appropriately interpreted. FS_{low} and FS_{high} denote the lower and upper bounds proved by Freitas and Siudeja in [18]. We know by comparison arguments, from [18], that FS_{low} improve F_{low}. Finally, HH_{high} is our new bound (5.2), while HH_{low} is the bound (6.8) derived by combining weighted Chiti and Crooke–Sperb inequalities; see Remark 6.3 below.

We are now ready to illustrate the various bounds by invoking the examples of two triangles, one of historical interest, and the other one is to prove and discuss what follows from this work and related earlier breakthroughs.

Remark 4.2. Consider a right isosceles triangle with equal sides of unit length. If the origin is taken at the midpoint of the hypotenuse, $\alpha = 1$. The Bessel zero $j_{1,1} \approx 3.83171$, and A_1 is explicitly given by

$$A_1 = A_{11} + A_{12}$$

where

$$A_{11} = \int_0^{\pi/2} \int_0^{\rho_1(\theta)} r^3 \sin^2\theta \mathrm{d}r\mathrm{d}\theta$$

and

$$A_{12} = \int_{\pi/2}^{\pi} \int_0^{\rho_2(\theta)} r^3 \sin^2\theta \mathrm{d}r\mathrm{d}\theta,$$

$$\rho_1(\theta) = \frac{1}{\sqrt{2}(\cos\theta + \sin\theta)}, \quad \rho_2(\theta) = \frac{1}{\sqrt{2}(-\cos\theta + \sin\theta)}.$$

Moreover,

$$B_1 = B_{11} + B_{12}$$

where

$$B_{11} = \int_0^{\pi/2} \left(1 + \frac{\rho_1'^2(\theta)}{\rho_1^2(\theta)}\right) \rho_1^2(\theta)\sin^2\theta d\theta$$

and

$$B_{12} = \int_{\pi/2}^{\pi} \left(1 + \frac{\rho_2'^2(\theta)}{\rho_2^2(\theta)}\right) \rho_2^2(\theta)\sin^2\theta \mathrm{d}\theta.$$

This leads to $A_1 = \frac{1}{24}$ and $B_1 = \frac{2}{3}$, and the bounds

$$45.0734 < \lambda < 58.7279.$$

If the right-angled corner is taken as the origin, $\alpha = 2$, and a similar calculation leads to

$$A_2 = \int_0^{\rho(\theta)} \int_0^{\pi/2} r^5 \sin^2(2\theta) \mathrm{d}r\mathrm{d}\theta = \frac{1}{45}$$

$$B_2 = \int_{\pi/2}^{\pi} \left(1 + \frac{\rho'^2(\theta)}{\rho^2(\theta)}\right) \rho^4(\theta) \sin^2(2\theta) \mathrm{d}\theta = \frac{4}{15}$$

where $\rho(\theta) = \frac{1}{\cos\theta + \sin\theta}$. With $j_{2,1} = 5.1356$, we obtain the two-sided bound

$$47.6325 < \lambda < 52.7492.$$

If one of the acute angled corners is chosen as the origin, $\alpha = 4$. With $\rho(\theta) = \frac{1}{\cos\theta}$, $j_{4,1} = 7.5883$, we obtain

$$A_4 = \frac{64}{525}$$

$$B_4 = \frac{128}{105}$$

and the bounds

$$45.9094 < \lambda < 57.5829.$$

Computations with this triangle for the various bounds found in the literature yield Table 1.

We note that the Faber–Krahn inequality yields a lower bound of 36.3368, and the Faber–Krahn for triangles leads to a lower bounds of 45.5858. For this triangle, $B = 6 + 4\sqrt{2}$ (see the table p. 257 in [33]). Thus, the upper bound provided by (1.5), with $j_{0,1} = 2.4048$ is 67.4136. Compare these numbers with the exact value of $\lambda = 5\pi^2 = 49.3506$ [33]. We note that Freitas [16] (see also [3]) proved the upper bound estimate for triangles

$$\lambda \leq \frac{\pi^2}{3A^2}\left(\ell_1^2 + \ell_2^2 + \ell_3^2\right) \tag{4.16}$$

which in our case results in an upper bound of $16\pi^2/3 \approx 52.6379$. Thus, the best quadrature for the fundamental eigenvalue of this triangle is provided by Payne–Weinberger's $\alpha = 2$ case and Siudeja's upper bound, i.e.,

$$47.6325 < \lambda < 51.1327.$$

Remark 4.3. We now systematically study the variation of these bounds as a function of α. We tabulate upper and lower bounds coming from the literature and our results for a right angle triangle with sides $a = \sin\pi/\alpha$, $b = \cos\pi/\alpha$ and $c = 1$. We place the origin of the coordinate system at the point with opening π/α. The comparison is summarized in Table 2 (see also Table 1). For all triangles, FS_{low} provides the best lower bound. As already noted, FS_{low} is a slight improvement on Freitas' earlier bound F_{low}. For bulky triangles, Siudeja's S_{high} gives the best bound, but as α increases, the best bound, denoted F'_{high}, is buried in the work of Freitas [15]. This bound was given up by Freitas for an esthetically appealing

Table 1. *Comparison of upper and lower bounds for a right isoceles triangle with unit sides*

α	S_{low}	F_{low}	FS_{low}	S_{high}	F_{high}	F'_{high}	FS_{high}	PS	HH_{high}	HH_{low}
4	28.7621	45.9094	**46.6702**	**51.1327**	54.8128	51.3066	169.241	67.4138	57.5829	20.41

The sharpest lower and upper bounds, to date, are in bold

Table 2. *Comparison of upper and lower bounds for a right triangle as a function of α: Siudeja (S_{low}, S_{high}) [38], Freitas (F_{low}, F_{high}, F'_{high}) [15], Freitas–Siudeja (FS_{low}, FS_{high}) [18], Payn–Weinberger (F_{low}) [15,30], Pólya–Szego (PS) [33], and Hasnaoui–Hermi (HH_{high} and HH_{low}, this article)*

α	S_{low}	F_{low}	FS_{low}	S_{high}	F_{high}	F'_{high}	FS_{high}	PS	HH_{high}	HH_{low}
4	57.524	91.819	**93.340**	**102.265**	109.626	102.613	111.238	134.828	115.166	35.536
6	73.668	120.369	**121.04**	130.965	127.928	**124.533**	150.040	172.665	131.635	59.944
8	105.017	166.873	**167.292**	186.697	171.833	**169.444**	219.060	246.143	175.095	94.002
10	145.918	224.803	**225.104**	259.409	228.582	**226.679**	308.148	342.007	231.662	137.030
12	195.398	292.766	**293.000**	347.374	295.863	**294.254**	415.421	457.981	298.844	188.844
14	253.157	370.237	**370.427**	450.058	372.886	**371.477**	540.223	593.36	375.819	249.353
16	319.072	456.946	**457.105**	567.239	459.275	**458.014**	682.237	747.853	462.174	318.502
18	393.082	552.731	**552.867**	698.812	554.819	**553.671**	841.282	921.319	557.694	396.247
20	475.154	657.479	**657.598**	844.718	659.378	**658.322**	1017.24	1113.68	662.237	482.556
22	565.25	771.108	**771.215**	1004.92	772.855	**771.874**	1210.01	1324.9	775.702	577.400
24	663.416	893.555	**893.651**	1179.41	895.176	**894.257**	1419.54	1554.94	898.014	680.754
26	769.586	1024.77	**1024.85**	1368.15	1026.28	**1025.42**	1645.78	1803.79	1029.11	792.597
28	883.774	1164.7	**1164.78**	1571.15	1166.13	**1165.31**	1888.68	2071.42	1168.95	912.911
30	1005.98	1313.32	**1313.39**	1788.40	1314.66	**1313.89**	2148.20	2357.84	1317.49	1041.68
32	1136.19	1470.59	**1470.66**	2019.89	1471.87	**1471.12**	2424.32	2663.04	1474.68	1178.88

The sharpest lower and upper bounds, to date, are in bold

form in Theorem 2 of [15] and is obtained by combining the formulas on pages 383 and 384 of his paper in the Rayleigh–Ritz characterization. It is explicitly given by the formula (our α corresponds to π/α in [15]):

$$F'_{\text{high}} = \frac{\frac{\pi}{\alpha}\int_0^1 \left(\frac{g'(\pi t/\alpha)}{g(\pi t/\alpha)}\right)^2 \sin^2(\pi t)\,\mathrm{d}t \int_0^{j_{\alpha,1}} z\,(J'_\alpha(z))^2\,\mathrm{d}z + \frac{\pi}{2\alpha}\int_0^{j_{\alpha,1}} zJ_\alpha^2(z)\mathrm{d}z}{\frac{\pi}{\alpha\, j_{\alpha,1}^2}\int_0^1 \frac{\sin^2(\pi t)}{g^2(\pi t/\alpha)}\,\mathrm{d}t \int_0^{j_{\alpha,1}} zJ_\alpha^2(z)\mathrm{d}z}$$

where $g(\theta) = \frac{\sin(\frac{\pi}{\alpha}-\theta)}{c\,\sin\pi/\alpha} + \frac{\sin\theta}{b\,\sin\pi/\alpha}$. The function $r = 1/g(\theta)$ describes the shorter length of the triangle in polar form. The test function choice $v(r,\theta)$ was optimized in [15] for the triangle and was given in terms of the Bessel function $J_\alpha(z)$

$$v(r,\theta) = J_\alpha(j_{\alpha,1}\, g(\theta) r)\sin\alpha\theta.$$

Our bound HH_{high} follows the Freitas form. However, our choice is optimized for a circular sector and is inspired by the Payne interpretation in Weinstein fractional space. Our new bound does not beat F_{high} or F'_{high}, but follows them, and is the next best bound, to date.

Remark 4.4. We next use the bounds in (4.15) to provide new lower and upper bound estimates for the fundamental eigenvalue of a regular α-polygon with n sides inscribed in a circular sector of radius 1 and opening π/α, for $\alpha = 1, 1.5, 2, 4, 3, 5, 6$ and $n = 2, 3, \ldots, 15$. Our example is inspired by the table p. 255 of [33]. The results for the α-polygons are summarized in Table 3. We note the clear monotonicity principle at play as we double the value of n. As $n \to \infty$, we expect the lower and upper bounds to converge to the fundamental eigenvalue of the circular sector of opening π/α, i.e., $j_{\alpha,1}^2$.

5. Lower bound for the relative torsional rigidity of a wedge-like domain

For wedge-like domains, relative torsional rigidity P_α was introduced in [20] where we proved its existence and fundamental properties. We also proved in [20] an isoperimetric Saint-Venant-type theorem which states that relative torsional rigidity is maximized for perfect sector with the same “weighted area” A_α

TABLE 3. *Lower and upper bounds for the fundamental eigenvalue of a regular α-polygon with n sides inscribed in a circular sector of radius 1 and opening π/α, for $\alpha = 1, 1.5, 2, 3, 4, 5, 6$ and $n = 2, 3, \ldots, 15$*

n	$\alpha = 1$		$\alpha = 1.5$		$\alpha = 2$		$\alpha = 3$		$\alpha = 4$		$\alpha = 5$		$\alpha = 6$	
2	18.401	29.364	20.640	26.921	25.533	30.900	38.493	43.629	54.473	59.861	73.098	78.869	94.236	100.437
3	16.145	19.576	20.380	22.866	26.001	28.268	39.718	41.972	56.194	58.584	75.224	77.779	96.724	99.482
4	15.585	17.201	20.408	21.640	26.284	27.418	40.285	41.412	56.952	58.142	77.416	76.142	97.784	99.150
5	15.315	16.232	20.397	21.103	26.389	27.036	40.520	41.156	57.273	57.940	77.244	76.533	98.237	98.997
6	15.153	15.736	20.371	20.819	26.425	26.832	40.623	41.018	57.420	57.830	77.150	76.715	98.451	98.914
7	15.046	15.447	20.345	20.649	26.435	26.709	40.672	40.935	57.493	57.764	77.094	76.808	98.561	98.864
8	14.972	15.293	20.322	20.541	26.435	26.630	40.696	40.881	57.533	57.722	77.058	76.859	98.622	98.832
9	14.918	15.138	20.302	20.467	26.431	26.577	40.709	40.845	57.555	57.693	77.033	76.889	98.658	98.810
10	14.879	15.050	20.287	20.414	26.427	26.538	40.715	40.818	57.568	57.672	76.907	77.015	98.681	98.794
11	14.847	14.986	20.274	20.375	26.422	26.509	40.718	40.799	57.576	57.656	76.919	77.002	98.695	98.782
12	14.823	14.936	20.263	20.345	26.417	26.488	40.720	40.784	57.581	57.645	76.926	76.992	98.705	98.773
13	14.804	14.898	20.254	20.322	26.413	26.471	40.720	40.773	57.584	57.636	76.931	76.984	98.711	98.766
14	14.788	14.868	20.247	20.304	26.409	26.458	40.720	40.763	57.586	57.628	76.934	76.978	98.718	98.761
15	14.776	14.844	20.241	20.289	26.406	26.447	40.720	40.756	57.587	57.622	76.937	76.973	98.719	98.756
$j_{\alpha,1}^2$ $(n \to \infty)$	14.6820		20.1907		26.3746		40.7065		57.5829		76.9389		98.7263	

[Ineq. (1.13) above]. Relative torsional rigidity lends itself also to the Payne interpretation in the proper Weinstein fractional space. This physical parameter P_α is defined via

$$\frac{1}{P_\alpha} = \inf\left\{ \frac{\int_D |\nabla f|^2 \,\mathrm{d}A}{\left(\int_D f h \,\mathrm{d}A\right)^2} : \; f \in H_0^1(D), f \not\equiv 0 \right\} \tag{5.1}$$

where as above $h = r^\alpha \sin \alpha\theta$. In fact, this Rayleigh–Ritz characterization is all we need to prove the following. Our new theorem provides a counterpart lower bound to the Saint-Venant inequality in the case of wedge-like domains defined by (1.1).

Theorem 5.1. *The relative torsional rigidity P_α of the wedge-like membrane D defined by* (1.1) *satisfies the inequality*

$$P_\alpha \geq \frac{A_\alpha^2}{(2\alpha + 4)B_\alpha}. \tag{5.2}$$

Equality holds if D is a sector of angle $\frac{\pi}{\alpha}$.

Remark 5.2. One can motivate this inequality in light of the higher-dimensional version of (1.6) found in [17] and the Payne interpretation in Weinstein fractional space, as done above.

Proof. The proof is again inspired by similar considerations in [33]. In the Rayleigh–Ritz principle (5.1), let $f = vh$ where

$$v(r, \theta) = g\left(\frac{r}{\rho(\theta)}\right), \tag{5.3}$$

and g is any C^1 function such that $g(1) = 0$. We calculate

$$\begin{aligned}
\int_D f h \mathrm{d}A &= \int_D v h^2 \mathrm{d}A \\
&= \int_0^{\frac{\pi}{\alpha}} \int_0^{\rho(\theta)} g\left(\frac{r}{\rho(\theta)}\right) r^{2\alpha+1} \sin^2 \alpha\theta \,\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{\alpha}} \left(\int_0^{\rho(\theta)} g\left(\frac{r}{\rho(\theta)}\right) r^{2\alpha+1}\mathrm{d}r \right) \sin^2 \alpha\theta \,\mathrm{d}\theta.
\end{aligned}$$

Let $s = \frac{r}{\rho(\theta)}$, then

$$\int\limits_D fh \mathrm{d}A = \int\limits_0^{\frac{\pi}{\alpha}} \rho^{2\alpha+2}(\theta) \sin^2 \alpha\theta \, \mathrm{d}\theta \int\limits_0^1 g(s) s^{2\alpha+1} \, \mathrm{d}s. \tag{5.4}$$

With A_α given in terms of $\rho(\theta)$ in (4.4), we finally have

$$\int\limits_D fh \mathrm{d}A = (2\alpha+2) A_\alpha \int\limits_0^1 g(s) s^{2\alpha+1} \, \mathrm{d}s. \tag{5.5}$$

For the Dirichlet integral of f, we have (4.9). Therefore, plugging (5.5) and (4.9) into the Rayleigh–Ritz principle (5.1), we obtain

$$\frac{1}{P_\alpha} \leq \frac{B_\alpha}{(2\alpha+2)^2 A_\alpha^2} \frac{\int_0^1 g'^2(s) s^{2\alpha+1} \, \mathrm{d}s}{\left(\int_0^1 g(s) s^{2\alpha+1} \, \mathrm{d}s\right)^2}. \tag{5.6}$$

Choose

$$g_0(s) = \frac{1-s^2}{4\alpha+4} \tag{5.7}$$

in (5.6) to match the radial part of the stress function corresponding to relative torsional rigidity for the perfect sector [see (5.25) in [20]], then

$$\frac{\int_0^1 g_0'^2(s) s^{2\alpha+1} \, \mathrm{d}s}{\left(\int_0^1 g_0(s) s^{2\alpha+1} \, \mathrm{d}s\right)^2} = 8(\alpha+1)^2(\alpha+2) \tag{5.8}$$

and

$$\frac{1}{P_\alpha} \leq \frac{(2\alpha+4)B_\alpha}{A_\alpha^2} \tag{5.9}$$

which is the stated result. To prove that the function in (5.7) is the minimizer of (5.6), and rather than appeal to the Euler-Lagrange equation, we proceed as in Pólya and Szegő's book [33] (see the argument pp. 91–93), via the Cauchy–Schwarz inequality. Since $g(1) = 0$ and g is C^1, we have

$$\begin{aligned}
\left(\int\limits_0^1 g(s) s^{2\alpha+1} \mathrm{d}s\right)^2 &= \left(-\int\limits_0^1 g'(s) s^{\frac{2\alpha+1}{2}} \frac{s^{\frac{2\alpha+3}{2}}}{2\alpha+2} \mathrm{d}s\right)^2 \\
&\leq \left(\int\limits_0^1 g'^2(s) s^{2\alpha+1} \mathrm{d}s\right) \left(\int\limits_0^1 \frac{s^{2\alpha+3}}{(2\alpha+2)^2} \mathrm{d}s\right) \\
&= \left(\int\limits_0^1 g'^2(s) s^{2\alpha+1} \mathrm{d}s\right) \frac{1}{(2\alpha+2)^2(2\alpha+4)}
\end{aligned}$$

Therefore,

$$8(\alpha+1)^2(\alpha+2) \leq \frac{\int_0^1 g_0'^2(s) s^{2\alpha+1} \, \mathrm{d}s}{\left(\int_0^1 g_0(s) s^{2\alpha+1} \, \mathrm{d}s\right)^2} \tag{5.10}$$

By (5.8) and (5.10), $g_0(s)$ is indeed the minimizer. $\square$

6. Rellich-type identity for a wedge-like domain and application

In this section, we show a version of the Rellich identity (1.9) for a wedge-like membrane. As an application, we obtain a counterpart of the Payne–Rayner inequality which is sharper than what Cauchy–Schwarz's inequality would naturally give. With the substitution $u = vh$ in (1.9), we are led to the following form of the Rellich identity which we will prove. We offer a proof from first principle though one can see in light of $\partial u/\partial n = h\,\partial v/\partial n$ and the fact that $v = 0$ on Γ that the result should hold directly from (1.9).

Lemma 6.1. *Let u be first eigenfunction of the Dirichlet problem in D. Then,*

$$\int_\Gamma (x,n)\left(\frac{\partial v}{\partial n}\right)^2 h^2\,d\sigma = 2\lambda\int_D u^2\,dA. \tag{6.1}$$

Proof. Plugging the substitution $u = v\,h$, $h = r^\alpha \sin\alpha\theta$ into the equation $\Delta u + \lambda u = 0$, we get

$$h\Delta v + 2(\nabla h, \nabla v) = -\lambda h v. \tag{6.2}$$

Multiplying that by $h(\nabla v, x)$ and integrating, we get

$$\int_D h^2\Delta v(\nabla v, x) + 2h(\nabla h,\nabla v)(\nabla v, x)\,\mathrm{d}A = -\lambda\int_D h^2 v(\nabla v, x)\,\mathrm{d}A \tag{6.3}$$

Now, an elementary calculation gives that

$$\int_D 2h(\nabla h,\nabla v)(\nabla v, x)\,\mathrm{d}A = \int_D (\nabla h^2,\nabla v)(\nabla v, x)\,\mathrm{d}A \tag{6.4}$$

$$= \int_D (\nabla h^2, (\nabla v, x)\nabla v)\,\mathrm{d}A. \tag{6.5}$$

Using the divergence theorem, we have

$$\begin{aligned}\int_D (\nabla h^2, (\nabla v, x)\nabla v)\,\mathrm{d}A &= \int_\Gamma (\nabla v, x)\frac{\partial v}{\partial n} h^2 \mathrm{d}\sigma\\ &\quad - \int_D (\nabla(\nabla v, x),\nabla v)\,h^2\,\mathrm{d}A\\ &\quad - \int_D (\nabla v, x)\Delta v\,h^2\,\mathrm{d}A.\end{aligned} \tag{6.6}$$

On the other hand, we have

$$\begin{aligned}\int_D (\nabla(\nabla v, x),\nabla v)\,h^2\,\mathrm{d}A &= \int_D \left((D(\nabla v).\nabla v, x) + |\nabla v|^2\right) h^2\,\mathrm{d}A\\ &= \frac{1}{2}\int_D (\nabla(|\nabla v|^2), x)\,h^2\,\mathrm{d}A + \int_D |\nabla v|^2 h^2\,\mathrm{d}A\end{aligned}$$

Using again the divergence theorem, we obtain

$$\begin{aligned}\int_D (\nabla(\nabla v, x),\nabla v)\,h^2\,\mathrm{d}A &= \frac{1}{2}\int_\Gamma |\nabla v|^2 (x,n) h^2\,\mathrm{d}\sigma\\ &\quad - \int_D |\nabla v|^2 h(\nabla h, x)\mathrm{d}A.\end{aligned}$$

Now, on $\Gamma = \partial D$, we have

$$(\nabla v, x) = ((\nabla v, n)n, x) = (x, n)\frac{\partial v}{\partial n}$$

Then

$$\int_D h^2 \Delta v(\nabla v, x) + 2h(\nabla h, \nabla v)(\nabla v, x)\ \mathrm{d}A = \frac{1}{2}\int_\Gamma \left(\frac{\partial v}{\partial n}\right)^2 (x,n)h^2 \mathrm{d}\sigma - \int_D |\nabla v|^2 h(\nabla h, x)\mathrm{d}A.$$

A similar calculation for the integral on the RHS of (6.3) gives

$$\begin{aligned}\int_D v(\nabla v, x)h^2\ \mathrm{d}A &= \frac{1}{2}\int_D (\nabla v^2, h^2 x)\ \mathrm{d}A \\ &= \int_\Gamma (x,n)v^2h^2\ \mathrm{d}\sigma - \int_D u^2\ \mathrm{d}A - \int_D v^2 h(\nabla h, x)\ \mathrm{d}A\end{aligned}$$

Now, using the fact that $(\nabla h, x) = \alpha h$, we get

$$\begin{aligned}\int_D |\nabla v|^2(\nabla h, x)h\ \mathrm{d}A &= \alpha\int_D |\nabla v|^2 h^2\ \mathrm{d}A \\ &= \alpha\lambda\int_D v^2h^2\ \mathrm{d}A \\ &= \lambda\int_D v^2 h(\nabla h, x)\ \mathrm{d}A.\end{aligned}$$

Finally, combining all these elements leads to the Rellich identity and completes the proof of the Lemma. □

As application, we show how to obtain at once the extension of the Crooke and Sperb [12] inequality (1.8) to the case of the wedge-like domain D.

Theorem 6.2. *Let u be the eigenfunction associated with the first eigenvalue λ of the fixed membrane problem on D defined by* (1.1)*. Then*

$$\left(\int_D uh\ dA\right)^2 \leq \frac{2B_\alpha}{\lambda}\int_D u^2\ dA. \tag{6.7}$$

Proof. Applying the divergence theorem to the vector $h^2\nabla v$ in D and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\left(\lambda\int_D vh^2\ \mathrm{d}A\right)^2 &= \left(\int_\Gamma \frac{\partial v}{\partial n}h^2\mathrm{d}\sigma\right)^2 \\ &\leq \int_\Gamma \left(\frac{\partial v}{\partial n}\right)^2 (x,n)h^2\mathrm{d}\sigma \ .\ \int_\Gamma \frac{1}{(x,n)}h^2\mathrm{d}\sigma\end{aligned}$$

Rellich identity's for the wedge-like domain D gives

$$\left(\lambda \int_D vh^2 \,\mathrm{d}A\right)^2 \leq 2B_\alpha \lambda \int_D u^2 \mathrm{d}A.$$

Substituting $u = v\,h$ leads to the statement of the Theorem 6.2. □

Remark 6.3. Combining (2.2) and (6.7), we obtain the new lower bound estimate for λ in terms of B_α which compete with existing results in the case D is a subset of the wedge $\mathcal{W}$.

$$\lambda \geq \left(\frac{\pi}{2\alpha}\right)^{1/(\alpha+1)} \frac{j_{\alpha,1}^{2\alpha/(\alpha+1)}}{B_\alpha^{1/(\alpha+1)}} \tag{6.8}$$

This is the wedge-like version of the isoperimetric inequality $B \geq 2\pi$ proved in [1]; see also the discussion in Crooke and Sperb's paper [12].

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