A HISTORY OF THE CLASSICAL ISOPERIMETRIC PROBLEM

BY

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Byrsa Tombs at Carthage and view of Goletta, Tunisia, circa 1899

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Byrsa tombs and surrounding, circa 1909

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ISOPERIMETRIC PROBLEM

Introduction. The famous old Isoperimetric Problem of the ancients was that of finding a simply closed curve of given length which incloses the largest area. Another problem closely related to this problem is that of finding among all curves which inclose a given area that one which has the shortest perimeter. It is easy to prove that the solution to either of the two problems leads logically to the solution of the other. The first problem may be stated analytically as that of finding an arc with equations in parametric form

$$x = x(t) , \quad y = y(t) , \quad t_1 \leqq t \leqq t_2$$

which satisfies the conditions

$$x(t_1) = x(t_2) \quad , \quad y(t_1) = y(t_3) ,$$

but does not otherwise intersect itself, which gives the length integral

$$\int_{t_1}^{t_2} \sqrt{x'^2 + y'^2}\, dt$$

a fixed value l, and maximizes the area integral

$$1/2 \int_{t_1}^{t_2} (xy' - yx')\,dt.$$

The solution is a circle. The general problem of the calculus of variations for which one integral is to be given a fixed value while another is to be made a maximum or minimum is called after this one an isoperimetric problem. Any problem where a fixed length is involved while an integral of any kind is to be made a maximum or minimum is called an isoperimetric problem. In this paper only the first two problems which are formulated above will be discussed.

The earliest attempted solution of the problem which has been preserved for us was that of Zenodorus. His solution was the one generally given for the problem until the time of Steiner. Steiner gave a very elegant and simple proof of a condition necessary for a solution but did not give a sufficiency proof. The first complete proof that the solution of the problem is a circle was given by Weierstrass. The earliest writers who attempted to solve the problem used a geometric method. Later complete theories have been given by the method of the calculus of variations, by the geometric method, and more recently by means of Fourier Series.

The discussion of the history of the Isoperimetric Problem is divided below into four sections. In Section I the origin of the problem and the proofs to the beginning of the nineteenth century are discussed. In Section 2 the development of the geometric proofs is traced from the beginning of the nineteenth century to the present. The greater part of Section 2 is concerned with the development of the theory of Steiner, and of the theory of parallel curves which was first applied to the solution of the Isoperimetric Problem by Minkowski. Section 3 consists of an outline of the proofs of the Isoperimetric Property of a circle which have been given by means of Fourier Series. In Section 4 solutions which have been given by means of the càlculus of variations are discussed. The greater part of Section 4 is a discussion of the application of the theory of Weierstrass to the Isoperimetric Problem.

The Bibliography which follows Section 4 is also divided into four groups to correspond to the four sections described in the preceding paragraph. References which merely mention the Isoperimetric Problem are not included.

The numbers in the text inclosed in parentheses and following an author's name, refer to the dates of birth and death of the author. The numbers inclosed in square brackets refer to the Bibliography.

1. Origin of the problem and early theories. The origin of the problem has been attributed to the Greeks but it is not known who among them was the first to state the problem or attempt a solution. The mathematical historians Montucla [4] and Cantor [3] quote a statement of Diogenes Laertius (third century) regarding Pythagoras (580?-500? B.C.) which has been interpreted to mean that Pythagoras knew the maximum property of the circle. The statement of Pythagorus is, however, not at all convincing as it merely asserts that of all plane figures the circle is the most beautiful.

The following remark [9] appears in the De Caelo of Aristotle (384-322 B.C.): "Now of lines which return upon themselves the line which bounds the circle is the shortest." Aristotle does not give any further explanation of the sentence and his subject is a philosophical rather than a mathematical one.

Archimedes (287-212 B.C.) may have been the first to attempt a mathematical solution of the problem. Simplicius who lived about the sixth century A.D. mentions a proof [14] of Archimedes and Zenodorus. However, Proclus (410-485) says [12] that later mathematicians arrived at a solution partly from the works of Euclid (450-374 B.C.) and partly from those of Archimedes. The mathematical historian Libri [6] gives a list of books found in the Cosmography of Maurolytus (1494-1575) and one called the Isoperimetric

Figures appears under the name of Archimedes. Isaac Barrow (1630-1677) in his preface [20] to the Book of Lemmas makes the following comment in a note on the margin of his book: "These men attribute to him (Archimedes) a book which is the Isoperimetry of Zenodorus, fragments of statics, and other works."

It is not clear whether the references of the last paragraph are to a work of Archimedes which has been lost or to the later collection of articles called the Book of Lemmas to which the name of Archimedes has by some critics thought to have been erroneously appended. Zenodorus in one of his proofs [16 Hultsch ed., vol. 3, p. 1194] used a theorem which he says was a theorem of Archimedes. This is evidence for a conclusion that Zenodorus was acquainted with the work of Archimedes, and that he would have mentioned the fact if Archimedes had arrived at a solution of the Isoperimetric Problem.

A statement [10] of the historian Polybius (201-120 B.C.) suggests that he may have known the maximum property of the circle. He says that most people judge the size of cities simply from their circumference and that when one tells them that a city or camp with a circumference of forty stades may be twice as large as one with a circumference of one hundred stades the statement seems astounding to them. He adds that the trouble is that we have forgotten our lessons in geometry.

The Greeks in general did not have a clear understanding regarding the relation of perimeter to area. Polybius adds to the above remark that not only ordinary men but also statesmen and military men judged the size of a camp by measuring the circumference. Proclus mentions [13] certain members of communistic societies who cheated their fellow members by giving them plots of land of larger perimeter but smaller area than the plots which they took for themselves. He says also that the theorem that all triangles formed on the same or equal bases, and always between the same two parallel lines are equal in area was regarded by the Greeks as paradoxical because the perimeter could be made as large as one pleased.

Zenodorus who lived probably during the period from 200 B.C. to 90 A.D. wrote a treatise on figures of equal perimeter. This work has been lost but has been partly preserved in the works [15,16] of Theon of Alexandria and Pappus who were contemporaries and probably lived during the fourth century A.D. Christopher Clavius (1537-1612) refers [17] to the works of Theon and Pappus as the sources from which he took the proofs which he gives regarding isoperimetric figures. Recently discussions of the theorems of Zenodorus have been given by Heath [7] and Chisini [8] but both works follow that of Pappus. The theorems of Zenodorus that are of interest for our problem are:

<u>Theorem I</u>. <u>Among all polygons of equal number of sides and equal perimeters, the regular polygon is greatest in area.</u>

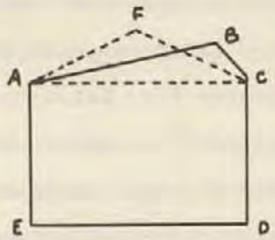
<u>Theorem II</u>. <u>The circle is greater in area than any regular polygon which has an equal perimeter.</u>

The proof of Theorèm I depends on two lemmas.

<u>Lemma I</u>. Among all triangles having the same base and the same sum of sides, the isosceles triangle is greatest in area.

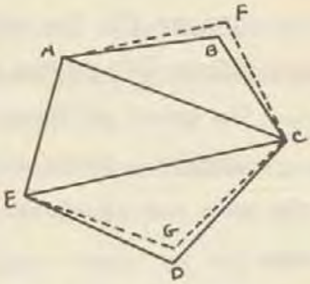
<u>Lemma II</u>. When two isosceles triangles are not similar to each other, if we construct on the same bases two triangles that are similar to each other, and such that the sum of the perimeters of the similar triangles is equal to that of the two original triangles, then the sum of the areas of the similar triangles is greater than the sum of the areas of the non-similar triangles.

The proof of Theorem I according to Zenodorus is as follows: Suppose AB and BC are unequal. Construct AF equal to CF and such that the sum of AF and CF is equal to the sum of AB and BC. Then by Lemma I, the triangle AFC is greater than the triangle ABC. Therefore the area

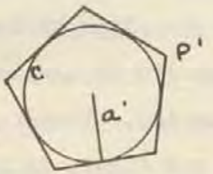


of the new figure is greater than the area of the original one, which is contrary to the hypothesis that the given figure be a maximum. Therefore the maximum polygon must be equilateral.

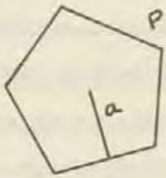
Now suppose the angle B is greater than the angle D, while the polygon is equilateral. Construct isosceles triangles AFC and EGC similar to each other and such that the sum of their perimeters is equal to the sum of the perimeters of ABC and EDC. Then by the use of Lemma II we know that the sum of the areas of the triangles AFC and EGC is greater than the sum of the areas of the triangles ABC and EDC. Therefore we have a new polygon of equal perimeter but greater area than the original one which is contrary to the hypothesis that the original polygon is a maximum polygon. By repeating the argument for other pairs of angles we finally conclude that the maximum polygon must be equiangular. Therefore the maximum polygon must be regular.



Zenodorus used the following demonstration as a proof of Theorem II. Let C be a circle of perimeter p, and P be a regular polygon of equal perimeter. Let P' be a polygon circumscribing C and similar to P. Let a and a' be the apothems of P and P', and



notice that a' is the radius of the circle. Since the polygons are similar we know that a/a' is equal to p/p'. But p' is greater than the perimeter p of the circle, therefore a' is greater than a. The area

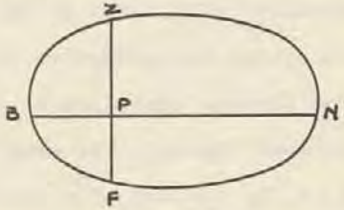


of C is, by the use of a theorem proved by Archimedes, equal to one-half the area of a rectangle the length of which is the perimeter and the width the radius of C, or a'p/2, and the area of P is ap/2. Therefore the area of C is greater than the area of P.

The two theorems when completely proved justify the conclusion that a circle of any given perimeter incloses a larger area than any polygon which has an equal perimeter. But in Theorem I it is assumed that among all polygons of a given perimeter there is one that is a maximum and this statement requires a proof. In addition we need for a complete proof of the Isoperimetric Theorem a discussion of the case when the figure which we are comparing with the circle is not a polygon.

Kepler (1571-1630) states the theorem [18] and refers the reader to the proof of Pappus. Galileo (1564-1642) gives an argument [19] which is essentially the proof of Zenodorus.

James Bernoulli (1654-1705) mentions the problem [21] and says that the solution is a circle but adds the statement needs to be proved. On the same page he proposed to mathematicians a more general problem. His problem is as follows:



Among all curves BFN of equal length and having a common base BN, to find one such that for a corresponding curve BZN whose ordinate PZ is any function of the ordinate PF or of the arc BF, the area BZNB shall be a maximum.

If we choose PZ equal to PF then the solution is two equal arcs of a circle. The solution will be a complete circle if we further choose the sum of the lengths of the two curves to be π times the length of BN.

The numerous attempts to solve the problem of Bernoulli and others similar to it led to the development of a new method for solving such problems. This new method Euler called the calculus of variations. All the early writers on the calculus of variations were concerned only with proving the conditions necessary for a solution. As a result the early proofs of the maximum property of the circle by the new method as well as the older geometric method were not complete.

Euler (1707-1783) developed a theory which is called the rule of Euler and applied it [22] to proving that if a curve of given length incloses a maximum area it is necessary that the curve be a circle. Riccati (1707-1775) has written a dissertation [26] in which he discusses the maximum property of the circle in a manner similar to that of Zenodorus which has been given in a preceding paragraph of this section. T. Simpson (1710-1761) gave a discussion [25] of the problem similar to that of Euler. He also gave a geometric demonstration [29] of the problem which is essentially the one given by Zenodorus. Other geometric proofs [23,30] much the same as that of Simpson have been given by Legendre (1752-1833) and P. Elvius (1710-1749). S. L'Huiler (1750-1840) proved [27] that if a curve which has a given length incloses a maximum area it is necessary that the radius of curvature be constant.

2. Geometric proofs of the isoperimetric property of the circle. The early attempts to prove the Isoperimetric Theorem by means of geometry lacked generality since only polygons were used as comparison curves. During the beginning of the nineteenth century an attempt was made to give a more general type of proof. The geometric proofs attempted from the beginning of the nineteenth century to the present can be roughly divided into two groups. The first group is largely synthetic and is closely associated with the work of Steiner (1796-1863). The second group is analytic and is concerned with proving an auxiliary theorem from which the Isoperimetric

Theorem follows at once. This auxiliary theorem says that for any closed curve the square of the perimeter minus 4 π times the inclosed area is greater than or equal to zero and the equality sign is valid only when the curve is a circle.

An attempt to prove that the area of a circle is larger than that of any closed curve of equal perimeter appeared in 1813 in an article [32] by an unknown author. The writer assumes that among all plane figures inclosing equal areas there is one that has the smallest perimeter. He then proves that this figure of smallest perimeter must be a circle. He argues that if one denies that it is a circle then some other figure must have this property. In that case he shows that a new figure can be constructed having equal area but smaller perimeter than the original one. But such a construction contradicts the hypothesis that the original figure possesses the smallest perimeter possible. Therefore, among all curves of equal area the circle has the shortest perimeter. Now suppose that there is a figure S having an equal perimeter and a larger area than a given circle C'. Make a circle C equal in area to S. The perimeter of C will be less than the perimeter of S by the preceding theorem. The area of C' will be larger than the area of C and therefore larger than the area of S which is contrary to hypothesis.

Steiner gave five different proofs [33] of the maximum property of the circle. His theorems include the analogous proposition that among all figures having equal areas a circle has the shortest perimeter. He states that each of these propositions implies the other as indicated for one of them in the last paragraph above, and therefore proves only one of them in any particular discussion. The proofs are general enough to include all closed curves having a given perimeter, but are in each case based on one of the following postulates which are not proved:

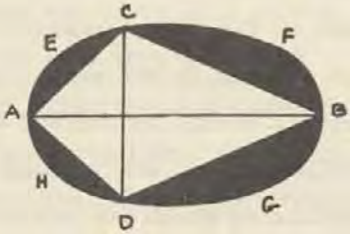
(1) Among all closed curves having a given perimeter there is at least one whose area is equal to or greater than the area of any of the others.

(2) Among all closed curves of given area there is at least one whose perimeter is less than or equal to the perimeter of any of the others.

Steiner comments on the first postulate as follows: "It is clear that there is an infinity of figures which have equal perimeters but which have different form and area. One observes that the area can be made as small as one pleases but not as large as one pleases, since all the figures may be inclosed in a circle whose radius is equal to one half the perimeter of the given figures and whose center is one point of the circumference of one of the figures. It must be that among these there is a maximum figure or several maximum figures, that is several figures which have equal perimeter but a larger area than any other figure not in the group."

<u>Theorem.</u> <u>Among all curves of equal perimeter the circle incloses the maximum area.</u>

Let EFGH be a maximum figure. One can find a line AB that divides the perimeter into two equal parts. Therefore, the area is divided into two equal parts because if not one can replace the smaller half by one equal to the larger



and thus increase the area of the original figure without changing the perimeter which is contrary to hypothesis. If the figure is not symmetric with respect to AB, replace one half of the figure by a figure symmetric to the other half, and since the area and perimeter are unchanged the figure is still one of the maximum figures. Now choose any point D on a semiperimeter. From it draw a perpendicular to AB and extend this perpendicular to meet the perimeter again at C, and draw the quadrilateral ACBD. The angles of the quadrilateral at C and D are right angles, because if not one can transform the figure so as to make them right angles while keeping the parts between the quadrilateral ACBD and the perimeter fixed, thus increasing the area without changing the perimeter which is contrary to hypothesis. Since D is an arbitrary point the figure must be a circle.

Since the other proofs are quite similar to the above, only a brief summary of the two proofs most discussed by writers following Steiner will be given. These are his second and fifth proofs. The second method introduces a quadrilateral drawn through any four points of the perimeter and the use of the fact that the maximum polygon of given perimeter is one that is inscribed in a circle in order to prove that for any figure other than a circle the area can be increased when the perimeter is kept fixed. The fifth method shows that a maximum figure must be symmetric to an arbitrary axis and therefore must be a circle.

Edler attempted to make a proof [34] by describing a geometric construction for making a regular polygon of at most 2^{n-1} sides from an irregular polygon of n sides. The constructed polygon has a smaller perimeter than the given polygon and a surface area at least equal to the surface area of the given polygon. He also proves that a circle has a larger surface area than any regular polygon which has an equal perimeter. These two theorems are combined into a proof of the maximum property of the circle as compared to polygons. To take care of the case when the figure is not a polygon the method of Steiner is used to increase the area while the perimeter remains constant. A polygon whose perimeter is equal to the perimeter of the original figure and which has also an equal area is inscribed in the increased figure. The theory regarding polygons is then applied to complete the proof.

Sturm (1841-1919) gave a discussion [35] of the maximum property of the circle. His proof of the condition necessary for a maximum is the same as that of Steiner. He has also simplified the construction of Edler described above.

Witting (1861-) proved [39] that any figure, such that if a line divides its perimeter into two equal parts, its area is also divided into two equal paets, is a figure with a center point. He completes the proof of the Isoperimetric Theorem by showing that a maximum figure must have such properties and therefore a center point and finally that all diameters are equal. His proof is open to the same objection as that of Steiner.

Padoa (1873-) has given an excellent review [40] of the first two proofs of Steiner.

Caratheodory (1873-) and Study (1862-1930) have written a joint paper [41] in which each gives a separate proof. They modify the method of Steiner so as to make a direct instead of an indirect proof. The change of method is made in order to avoid the necessity of making the assumption that a maximum exists.

An outline of the proof of Caratheodory is as follows: The plane is divided into halves by a line g, and from a point A on this line an arbitrary curve of length π is constructed so that its end point is also on g. A modification of the method of Steiner is used to construct an infinite series of auxiliary curves all of length π and having a common beginning point A, but whose end points are a series of points on g. Since these end points are at most a distance π from A, it is possible to select a subset which has one and only one limit point ω. The areas inclosed between the curves and the line g are denoted by

$$I_1, \quad I_2, \quad I_3, \text{ ------- } \quad I_n, \text{ ----}$$

and the method of construction shows that no one is smaller in area than the one preceding. Since the areas of these curves have an upper bound and are increasing they must converge to a limit, and Caratheodory proves that these curves have as a limiting curve the semicircle drawn through A and ω. Since the lengths of the curves remain equal to π, the circle must have a radius equal to one. Finally if I_0 denotes the area inclosed between the first arbitrarily selected curve and the line g, then

$$I_0 = I_n = \pi/2 \ ,$$

and the equality sign can hold only when I_0 is a semicircle, which proves that a semicircle of length π incloses a larger area with g than any other curve of equal length.

Study makes use of the symmetric method of Steiner. He starts with a polygon which has a perimeter 2π in length and shows how to construct a new convex polygon that has n axes of symmetry. For the new polygon the perimeter remains constant and the area is not less than the original one. As n increases there is produced a series of polygons that approach as a limit a circle whose radius r is to be determined. Since the perimeters of the polygons remain equal to 2π and are either equal to or greater than the perimeter of the limiting circle, it follows that $r \leqq 1$. Furthermore the areas of the polygons must approach the area πr^2, and hence the area of the originally given polygon must be $\leqq \pi$. If the equality sign holds, all the polygons of the series must be equal in area. But in that case it is pointed out that the method of construction leads to a contradiction. It follows that a circle of perimeter 2π has a larger area than any polygon of equal perimeter.

E. De Leber has given three proofs [42] based again on the postulate that there is a maximum figure among all those figures having a given perimeter.

Bieberbach (1886-) has given a proof of the theorem [44] that among all domains that are plane, finite, and have a given diameter the circle has the largest area, where he means by diameter the maximum distance between any two points of the border.

Minkowski (1864-1909) developed a theory [36] regarding pairs of parallel ovals that reduces to the Isoperimetric Theorem for a special case. This work is the beginning of the second group of proofs mentioned in the first paragraph of this section. He defines an oval by means of a function $H(u,v)$ which appears in the inequality,

$$ux + vy \leqq H(u,v).$$

The totality of such inequalities for all possible values of u and v defines a domain of points x,y which is called an oval. The area of an oval is

$$1/2 \int_0^{2\pi} H(H + \frac{\delta^2 H}{\delta \theta})\, d\theta$$

where H is defined as follows: Draw a unit circle about the center of gravity of the oval and consider this center of gravity as the origin of the system of coordinates. For some point α, β on the perimeter of the unit circle write $\alpha = \mathrm{Cos}\,\theta$, $\beta = \mathrm{Sin}\,\theta$. Choose the point on the boundary of the oval where the outer normal has the direction α, β and write

$$H(\alpha,\beta) = H(\cos\theta, \sin\theta) = H.$$

Suppose that we have two such ovals defined by H_1 and H_2 and construct a third such that $H = (1-t)H_1 + tH_2$. If we denote the area of this third oval by F then

$$F = (1-t)^2 F_1 + 2(1-t)tM + t^2 F_2$$

where F_1 and F_2 are the areas of the first two ovals and M is given by the equation,

$$M = 1/2 \int_0^{2\pi} H_1 (H_2 + \frac{\delta^2 H_2}{\delta \theta^2})\, d\theta = 1/2 \int_0^{2\pi} H_2 (H_1 + \frac{\delta^2 H_1}{\delta \theta^2})\, d\theta.$$

This quantity M is an invariant with respect to all parallel translations. Minkowski calls M the mixed area of the two curves. It is proved that

$$(1)\quad M^2 - F_1 F_2 \geqq 0$$

and that the equality sign is valid only when the two ovals are similar (homothetic). If the second oval is chosen as a unit circle, H_2 becomes the integer one and the expression for M shows that 2M is the perimeter of the first oval. If the perimeter and area of the first oval are denoted by L and F then (1) gives

$$L^2 - 4\pi F \geqq 0$$

and the equality sign is valid only when the first oval is a circle. It is easily verified that this last sentence is a statement of the isoperimetric property of the circle.

The theory of parallel curves has been discussed by several writers. Crone (1877-) has an article [37] on this subject that appeared about one year after the work of Minkowski. Blaschke (1885-) gave a discussion [45] of the theory of Minkowski in an article that appeared in 1914. He discussed first the case when two quadrilaterals take the place of the ovals of Minkowski. The theory is then extended to polygons and finally to any closed curves. A paper [46] written by Frobenius (1849-1917) was published in 1915 and another [49] by Liebmann (1874-) in 1919. The last two papers use the theory of quadratic forms to prove the isoperimetric property. If the area F(t) of any curve of a family is expressed in terms of the area F and the perimeter L of an arbitrary curve, for example by a formula

$$F(t) = F + tL + \pi t^2,$$

then $F(t)$ can be made negative for real values of t only when the discriminant of the quadratic form is greater than zero. The motive is then to show that for any closed curve that is not a circle a value of t exists for which $F(t)$ is negative and therefore the discriminant is greater than zero. The greater part of Liebmann's paper is concerned with proving this last statement. Bonnesen (1873-) has given an account of the theory of the mixed area of two ovals in his book [53] published in 1929. Blaschke gave a proof [55] of the Isoperimetric Theorem in his Differential Geometry. His proof is similar to that of Crone and Frobenius, and is notable for its simplicity. It depends on a theorem as follows:

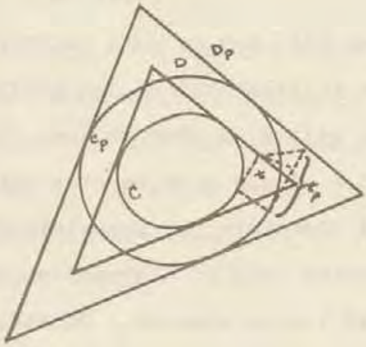
Theorem. There exists between the area F and the perimeter L of a circle the relation

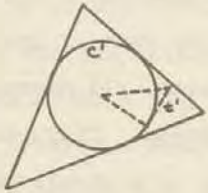
$$L^2 - 4\pi F = 0$$

and for every other plane curve the relation is

$$L^2 - 4\pi F > 0.$$

Let C be an oval with the clockwise direction prescribed on it, and let C_p be the outer parallel curve at a distance p with the tangents to C_p parallel in the same sense to the tangents of C. C_p is evidently an oval. Let C' be a unit circle. Let a triangle of tangents to C be denoted by D whose area contains C. Let D_p and D' be the corresponding triangles of parallel tangents for C_p and C'. Let t be the length of a line measured in a positive sense along a tangent from a point of contact on C to the intersection of this tangent with a side of D. Let t_p and t' be the corresponding lengths for parallel tangents to C_p and C'. Let ϕ be the angle between the tangent and a fixed direction. Let d, d_p, and d' be the areas of D, D_p, and D'. Let F, F_p, and F' be the areas of C, C_p, and C'.





Let r be the radius of the inscribed circle to the triangle D. Then we have the following relations,

$$t_p(\phi) = t(\phi) + pt'(\phi) \;,\; F_p = d_p - 1/2\int_{\pi}^{\pi} t_p^2 \, d\phi \;;$$

$$(1) \quad F_p = (p + r)^2 d' - 1/2\int_{-\pi}^{\pi} (t + p\,t')^2 \, d\phi .$$

The area of the parallel curve may be expressed by the equation

$$(2) \quad F_p = F + pL + p^2F'$$

where L denotes the perimeter of C. This can be seen as follows: Consider the small trapezoid inclosed by the arc length ds of C, the two lengths measured along the normals to C through the ends of ds, and the arc of the outer curve intercepted by these lengths. The area of each of these small figures is

$$p(ds + 1/2\ p\ d\phi)$$

and therefore by taking the sum of them one sees that

$$F_p - F = \int p(ds + 1/2\ p\ d\phi) = pL + p^2\pi$$

For $p = 0, F_p$ is positive by equation (2) and for $P + r = 0$ is less than or equal to zero by equation (1). Therefore the roots of $F_p = 0$ are real and it follows that

$$L^2 - 4\pi F \geq 0$$

If the equality sign holds F_p must not change in sign. It is positive for p = o, therefore if one sets p =-p in (1) the integral must vanish and the ratio t/t' equal to r is constant. Therefore one can vary one side of the triangle of tangents and r will remain unchanged. It follows that all circumscribed triangles to the oval C have an inscribed circle such that the diameters of all the circles are equal. A figure such that all its diameters are equal must be a circle. The final remark of the author is that the proof is valid only when the perimeter of the figure contains no corners or straight lines.

Lebesgue (1875-) wrote a paper [43] in 1914 in which he shows that the Isoperimetric Problem may be stated in a new way. His statement of the problem is as follows: Find a domain for which the ratio L^2/F, the square of the perimeter to the area, shall be the smallest possible.

Bláschke has given a proof of the Isoperimetric Theorem by means of a method which is different from that described above. He has written two articles in which the same method is used. His paper [45] of 1915 is a brief account of a proof which is given more in detail in his book [47] which was published in 1916. It is proved that if one has a closed, continuous, rectifiable, plane curve of length L and area F one can construct a polygon such that if λ is its perimeter and ϕ its area then it is true that

$$|L - \lambda| < \epsilon, \ |F - \phi| < \epsilon$$

where ϵ is an arbitrary quantity, and it also follows that

$$\lambda^2 - 4\pi\phi > 0$$

Now suppose that

$$L^2 - 4\pi F < 0$$

then one can construct a polygon such that

$$\lambda^2 - 4\pi\phi < 0$$

which is contrary to the results of a previous proof and therefore one must have

$$L^2 - 4\pi F \geqq 0$$

The method of Steiner may now be used to show that the equality sign holds only for a circle.

Bonnesen has written a number of articles [50,51,52] on the Isoperimetric Problem but he has given the essentials of all this material in a book [53]. He begins the discussion with the remark that it is sufficient to prove the Isoperimetric

Theorem for convex curves because it is easy to see that a curve which is not convex can not furnish a maximum. He calls the expression $L^2/4\pi - F$ the "Isoperimetric Deficit" of a curve. He states that the formulas

$$L_1 = L + 2\pi t \quad , \quad F_1 = F + Lt + \pi t^2$$

can be easily derived for the case of two parallel polygons, and for the general case it is sufficient to pass to the limit. He proves that

$$L^2/4\pi - F \geqq (\pi/4)(R-r)^2$$

for a convex curve, where R and r are the radii of the circumscribed and inscribed circles of the curve. Bernstein (1878-) in 1905 worked out a number of inequalities for curves on a sphere. One of the relations which he obtained for the case of a plane is the inequality of Bonnesen which has been written above except that the constant factor $\pi/4$ is replaced by $1/8(1 + 2\pi)^2$. Bonnesen refers to this inequality of Bernstein and says that it was the first inequality of this type to be derived.

A brief outline of the proof of Bonnesen is as follows: Consider the quadratic form

$$F + Lt + \pi t^2.$$

Its discriminant is precisely the deficit of the curve when one divides out the factor 4π. The deficit is greater than or equal to zero when the roots of the quadratic form are real. The form is positive for $t = 0$. If there exist values of t for which the form is less than or equal to zero then the roots are real. If there exists a value of t for which

$$Lt - F - \pi t^2 \geqq 0,$$

then the negative of this value of t will make

$$F + Lt + \pi t^2 \leqq 0.$$

The proof is complete when it is proved that

$$Lt - F - \pi t^2 \geqq 0$$

for t equal to r or R, the radii of the inscribed and circumscribed circles to the convex figure.

Bonnesen proves the last statement in the preceding paragraph when the figure is a convex polygon as follows: Inscribe a circle in the polygon touching two sides, and if the sides are parallel this is the largest circle that can be inscribed, but if not one can always draw the circle to touch three sides. Draw tangents to the circle at the three points where it touches the polygon. These tangents will form a triangle. Move each side of the polygon parallel to a fixed direction, say parallel to the bisectors of the angles of the triangle or parallel to any two parallel sides of the polygon until each side of the polygon is tangent to the circle. If a is the length of a side of the polygon, H its distance from the center of the circle,

and r the radius of the circle then one has

$$\Sigma [1/2 Ha - (H - r)a] = (\Sigma ra - \Sigma 1/2 Ha) \geqq \pi r^2$$

But

$$F = \Sigma 1/2 Ha, \quad L = \Sigma a$$

and therefore

$$(1) \quad rL - F \geqq \pi r^2$$

and the equality sign is valid only when the figure is a circle. But (1) can be written in the form

$$(2) \quad L^2/4\pi - F \geqq \pi(L/2\pi - r)^2$$

When a similar process is carried out for a circumscribed circle another inequality is produced, namely,

$$(3) \quad L^2/4\pi - F \geqq \pi(R - L/2\pi)^2$$

The inequalities (2) and (3) may be combined into the inequality

$$(4) \quad L^2/4\pi - F \geqq \pi/4(R - r)^2$$

The Isoperimetric Theorem can be easily deduced from this result.

Bonnesen says that the method can be applied to any convex curve by moving all tangents and all straight lines of the perimeter parallel to a fixed direction as in the case of the polygon. He has given a second proof [53] by means of a symmetric method. He has also modified the symmetric method of Steiner. In each case he arrives at the result (4) given in the preceding paragraph.

3. Proofs by means of Fourier Series. The proof of the Isoperimetric Theorem by means of Fourier Series depends on a theorem which Hurwitz (1859-1919) called the fundamental theorem. This theorem is as follows: If

$$F(u) \sim a_0/2 + \sum_1^{\infty}(a_k \text{Cos } k\, u + a'_k \sin k\, u)$$

and

$$H(u) \sim b_0/2 + \sum_1^{\infty}(b_k \text{Cos } k\, u + b'_k \sin k\, u)$$

then

$$1/\pi \int_0^{2\pi} F(u)H(u)d\, u = a_0 b_0/2 + \sum_1^{\infty}(a_k b_k + a'_k b'_k)$$

where the sign $\sim$ is to be read equivalent to, and means that the right hand side of the first two equivalences represent the usual trigonometric development of the functions except that nothing is implied as to the convergence of the series or its equality with the function on the left. The equivalent sign may be replaced by the equality sign when the function defining the series has suitable continuity properties.

Let $x = x(s)$, $y = y(s)$, $0 \leqq s \leqq L$ be the equations of a simply closed, continuous, rectifiable curve where s is the length of arc and L is the length of the perimeter. It is known that a function which is continuous, periodic, and has a derivative which is continuous except at a finite number of finite discontinuities, may be expressed by a Fourier Series which converges uniformly and absolutely. Let us suppose that the functions for x and y which define the curve possess these propert'es. Then if we substitute for s the parameter u given by the equation $u = 2\pi S/L$. x and y may be expressed as follows:

$$x = a_0/2 + \sum_1^\infty (a_K \operatorname{Cos} k u + a'_K \sin k u),$$
$$y = b_0/2 + \sum_1^\infty (b_K \operatorname{Cos} k u + b'_K \sin k u).$$

The derivatives with respect to u are:

$$dx/du \sim \sum_1^\infty k(a'_K \operatorname{Cos} k u - a_K \sin k u),$$
$$dy/du \sim \sum_1^\infty k(b'_K \operatorname{Cos} k u - b_K \sin k u).$$

When the variable s in the equation $(dx/ds)^2 + (dy/ds)^2 = 1$ is replaced by u the equation becomes

$$(dx/du)^2 + (dy/du)^2 = (L/2\pi)^2.$$

The integration of this equation gives

$$\int_0^{2\pi} \left[(dx/du)^2 + (dy/du)^2\right] du = 2\pi(L/2\pi)^2 = L^2/2\pi.$$

The application of the fundamental theorem described above to this last equation gives

$$(1) \quad \pi \sum_1^\infty k\,(a_K^2 + a_K'^2 + b_K^2 + b_K'^2) = L^2/2\pi$$

The area inclosed by the curve may be represented by the equation

$$F = \int_0^{2\pi} (x\,dy/du)\,du.$$

But again on account of the fundamental theorem this reduces to

$$(2) \quad F = \pi \sum_1^\infty k(a_K b'_K - a'_K b_K).$$

It follows from (1) and (2) that

$$L^2/2\pi - 2F = \pi \sum_1^\infty \left[(ka_K - b'_K)^2 + (ka'_K + b_K)^2 + (k^2-1)(b_K^2 + b_K'^2)\right],$$

and since the right hand side of this equation is never negative it must be true that

$$L^2 - 4\pi F \geq 0$$

One observes that the equality sign is valid only when

$$a'_1 + b_1 = 0,\ a_1 - b'_1 = 0,\ a_k = a'_k = b_k = b'_k = 0 \quad (k = 2,3,4----).$$

But then the series for x and y reduce to

$$x = a_0/2 + a_1 \operatorname{Cos} u + a'_1 \sin u,$$

$$y = b_0/2 - a_1 \operatorname{Cos} u + a_1 \sin u$$

which are the parametric equations of a circle. Therefore for all simply closed, rectifiable, plane curves whose equations $x = x(s)$, $y = y(s)$ have the properties described at the beginning of this paragraph it is true that

$$L^2 - 4\pi F \geqq 0$$

and the equality sign is valid only when the curve is a circle. The Isoperimetric Theorem follows easily from this inequality.

The first proof [56] of the Isoperimetric Theorem which made use of the Fourier Series was given by Hurwitz. Lebesgue has also given a proof [57] similar to that given by Hurwitz. Each of these proofs is preceded by a detailed discussion of the theory of the Fourier Series. The application of the theory to the proof of the Isoperimetric Theorem is in each case the same except for a few details. The type of curves to which the proofs apply is determined by the development of the theory of the Fourier Series. Hirschler has written a thesis [58] in which the area of a regular closed curve is compared with that of a circle by means of the Fourier Series. He defines a regular curve as follows: A regular curve is continuous, consists of a finite number of arcs which do not intersect themselves, and each arc possesses a continuously turning tangent at every interior point and end point. He applies the fundamental theorem which is stated in the first paragraph of this section to the symmetric formula for the area

$$A = 1/2\int_0^{2\pi}(xy' - yx')\,d\,S,$$

and uses the integral of the relation

$$(dx/ds)^2 + (dy/ds)^2 = 1$$

to compute the area of a circle which has a perimeter equal to the perimeter of the regular curve in terms of the Fourier Constants of $x(s)$ and $y(s)$. He proves that the difference between the area of a circle of given perimeter and the area of a regular curve which has an equal perimeter is a positive quantity unless the regular curve is itself a circle. The proof is again preceded by a development of the Fourier Series. A good outline [55] of the proof of Hurwitz is given by Blaschke in his Vorlesungen über Differential Geometrie.

4. Proofs by means of the calculus of variations. A statement was made in Section 1 that the early proofs of the Isoperimetric Theorem which applied the calculus of variations were concerned only with the necessary condition for a maximum. This statement continues to be true for all such proofs which were published before the work of Weierstrass (1815-1897) was known. Many of the writers discussed a modification of our problem which we will call Problem I, and of which our problem

is a special case.

Problem I. To find among all curves which have a given length and join two fixed points on a straight line that one which incloses with the line a maximum area.

A first necessary condition that a simply closed, regular curve of given length inclose a maximum area, if such a curve exists, may be derived from the theory of Problem I by assuming that the two points on the line coincide. A first necessary condition on a solution of Problem I is that the maximizing curve shall satisfy the first necessary conditions that the integral

$$(1) \qquad \int_{t_1}^{t_2} \left[1/2(xy' - yx') + \lambda \sqrt{x'^2 + y'^2} \right] dt$$

have a maximum value where λ is a suitably selected constant. If we write H for the integrand of the integral, these conditions are that the derivatives $H_{x'}$, $H_{y'}$, be continuous on the maximizing arc and that the differential equations of Euler must be satisfied. The conditions that $H_{x'}$, and $H_{y'}$, are continuous imply that the maximizing arc has no corners. The differential equations of Euler are

$$(2) \qquad \begin{aligned} H_x - \tfrac{d}{dt} H_{x'} &= y' - \lambda \tfrac{d}{dt}\left(x'/\sqrt{x'^2 + y'^2} \right) = 0, \\ H_y - \tfrac{d}{dt} H_{y'} &= -x' - \lambda \tfrac{d}{dt}\left(y'/\sqrt{x'^2 + y'^2} \right) = 0. \end{aligned}$$

When they are integrated we find

$$y - b = \lambda y'/\sqrt{x'^2 + y'^2}, \quad x - a = -\lambda x'/\sqrt{x'^2 + y'^2},$$
$$(x-a)^2 + (y-b)^2 = \lambda^2 .$$

Therefore the solution must be an arc of a circle. It is clear that the method is the same when the two points on the line coincide.

Proofs arriving essentially at the result above have been given by Bordoni (1789-1860) [59], Dienger (1818-1894) [63], Jellett (1817-1888) [60], Moigno-Lindelof [61], and Lundstrom [62], and more recently by Thomé (1841-1910) [68], and Hadamard (1865-) [70].

Weierstrass was the first to make a complete proof by the method of the calculus of variations that the area inclosed by a circle is larger than that inclosed by any other regular closed curve of equal length. His proof also precedes the other complete proofs by other methods. Unfortunately he did not publish his work. The record of it was preserved however, by means of the lecture notes of his students. The seventh volume of his collected works [74] based on several collections of these notes, was published in 1927.

Schwarz (1843-1921) used a method [64] similar to the method of Weierstrass to show that among all closed curves which inclose a given area the circle has the shortest perimeter. It was pointed out in Section 2 that the Isoperimetric Theorem follows from this result.

Kneser (1862-1930) has a discussion of the Isoperimetric Theorem in his book [65] on the calculus of variations. He develops both the necessary and the sufficient conditions for a maximum for Problem I. In the second edition of his book he deduces the Isoperimetric Theorem from the solution for Problem I. His argument is as follows: Consider a closed curve consisting of a finite number of regular parts, and let the length of the curve be l. Find two points P_1, P_2 which divide the perimeter into two equal parts and join these points with a straight line. Because of the solution to Problem I, the points may be joined by arcs of circles each of length l/2 in a manner such that the area inclosed by the two arcs is greater than the area inclosed by the original curve. It remains to be proved that a circle which has a circumference equal to l incloses a larger area than the area inclosed by the two arcs of circles the sum of whose lengths is l, and which join the two points P_1, P_2. Kneser does this by proving that a semicircle which has a length l/2 incloses a larger area with its chord than any other arc of length l/2 incloses with the same chord. To prove this consider an arbitrary arc of a circle which has length l/2 and join the ends of this arc by a line. Draw the radii of the arc at the two ends. Call the central angle between these two radii θ and denote the area between the arc and its chord by F. Then the area F

may be expressed by the equation

$$F = \theta/2(\ell/\theta)^2 - (\ell/\theta)^2 \sin\theta/2 \cos\theta/2 = \left(\frac{\theta - \sin\theta}{\theta^2}\right).$$

Take the derivative of F with respect to θ, set this derivative equal to zero, and divide out the constant factor $1^2/8$. This gives

$$\frac{-\theta^2 - \theta^2\cos\theta + 2\theta\sin\theta}{\theta^4} = \frac{-2\theta^2\cos^2\theta/2 + 4\theta\sin\theta/2\cos\theta/2}{\theta^4}$$

$$= 4/\theta \cos\theta/2(\sin\theta/2 - \theta/2\cos\theta/2) = 0.$$

This has only one solution π between 0 and 2π and this solution gives a maximum. But when the central angle is equal to π the figure is a semicircle. Therefore a semicircle incloses with its chord a larger area than any other arc of equal length. It follows that a circle constructed from two such semicircles possesses a perimeter equal to the sum of the lengths of the two arcs joining the two points on the original curve and an area greater than the area included between the two arcs.

Hancock (1867-) gave a discussion of the theory of Weierstrass in a book on the calculus of variations which was published in 1904. He gave a proof [67] of both the necessary and sufficient conditions for a closed curve to inclose a maximum area.

Bolza (1857-) has given an excellent discussion [69] of Problem I. His derivation of the first necessary condition is different from that given in the third paragraph of this section. He reduces the differential equations of Euler to the equivalent form

$$H_{xy'} - H_{yx'} + H_1(x' y'' - y' x'') = 0,$$

where

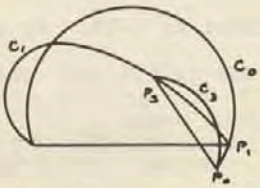
$$H_1 = \lambda/(\sqrt{x'^2 + y'^2})^3$$

and therefore the differential equation reduces to

$$1/r = (x' y'' - y' x'')/(\sqrt{x'^2 + y'^2})^3 = -1/\lambda.$$

This shows that λ is different from zero and that the curve sought must be an arc of a circle.

The sufficiency proof of Bolza for Problem I is as follows: Draw a circular arc C_0 of length 21 through the points P_1, P_2. Near this draw an arbitrary, admissible curve C' of the same length through the same two points. Let P_0 be a point on the extension of C_0 and suppose that it is not a point of C'. Choose an arbitrary point P_3 on C'. Then P_3 is different from P_0 and the sum Z_3 of the length of the arcs P_0P_1 plus P_1P_3 is larger than the distance P_0P_3. There can be drawn only one arc of a circle C_3 through P_0 and P_3 which has a length equal to Z_3 and which is described in a positive sense. Bolza has shown in a previous paragraph that a congruence of space extremals through P_0 may be defined by the equations



$$x-x_0 = -2\lambda \cos(T+K)\sin T, y-y_0 = -2\lambda \sin(T+K)\sin T, Z = -2\lambda T,$$

where T, λ, K are limited to the domain,

$$0 < T < \pi \quad , \quad 0 \leq K < 2\pi \quad , \quad \lambda < 0.$$

This congruence makes a space field S which on account of the inequality

$$z_3 > \sqrt{(x - x_0)^2 + (y - y_0)^2} > 0$$

fills up a definite part of space and C' lies wholly in this field. The E-Function of Weierstrass for this problem is

$$\mathcal{E}(x_3, y_3; p_3, q_3; p_3, q_3; \lambda_3) = \lambda_3[1 - \cos(\theta_3' - \theta_3)]$$

where λ_3 is negative and $\cos(\theta_3' - \theta_3)$ is not zero along the whole curve C', since we suppose that C' is different from C_0. The points conjugate to P_0 may be expressed by the zero values of a certain determinant. This determinant for this problem may be reduced to

$$\Delta(T_3, K_3, \lambda_3) = 8\lambda_3^2 \sin T_3(\sin T_3 - T_3 \cos T_3).$$

It is clear that the determinant is not equal to zero when

$$0 < T_3 < \pi, \quad \lambda_3 < 0.$$

Therefore it follows from the theory of Weierstrass that the integral (I) along the arc of the circle C_0 is greater than the same integral along any other curve C' which is admissible and different from C_0 This says that the solution to Problem I is an arc of a circle.

Bolza does not extend the method so that it applies to a complete circle but states that such an extension can be made.

After one has found the conditions necessary that a curve inclose a larger area than any other curve of equal length the proof may be made complete by proving that these conditions, or a modification of them, are sufficient to insure a maximum, or one may use a second method which consists in proving that there exists a curve of given length which incloses a maximum area. Tonelli (1885-) has used the second method [72] in proving the Isoperimetric Theorem.

Bonnesen devotes a chapter in his book [75] to the proof of the Isoperimetric Theorem. He uses polar tangential coordinates and the method of Weierstrass. He states his results as follows: Let E be a circle of radius r, (T) the complete ensemble of triangles and pairs of parallel lines circumscribed to E. Let C be the ensemble of convex curves which may be inscribed in one of the figures T. Between the perimeter L and the area F of any one of these curves C one has the inequality

$$rL - F \geqq \pi r^2$$

and the equality sign is valid only when C is a circle. This agrees with his result which has been described in Section 2.

Another problem closely related to the Isoperimetric Theorem is the problem which has been called the problem of Dido. It may be stated as follows: To find among all curves of given length which join two points on an arbitrary curve one which incloses with the arbitrary curve a maximum area. If the arbitrary curve is a straight line and the points are fixed it reduces to Problem I, but if the end points are variable it reduces to a new problem. It is known that if the end points are variable the maximum curve must intersect the arbitrary curve at right angles. If the arbitrary curve is a straight line the maximum curve is a semicircle. It is easy to deduce the Isoperimetric Theorem from this result. Kneser has made an extensive study [66] of the problem for the case of variable end points. Merrill (1887-) has also written a paper [71] on the variable end point case. There is also a paper by G. Weyl [73] in which sufficient conditions for a more general problem of the calculus of variations are applied to the problem of Dido.

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