PHY 3513 — Solutions —

Test III – 18 Nov 2016

(1) (a) $Z = g_1 + g_2 e^{-\varepsilon_2/k_B T} + g_3 e^{-\varepsilon_3/k_B T}$ 8/

$g_1 = 1$, $g_2 = 3$, $g_3 = 5$ and $\varepsilon_2 = k_B T_0$, $\varepsilon_3 = 2 k_B T_0$

$\Rightarrow \boxed{Z = 1 + 3e^{-1} + 5e^{-2} = 2.780}$ at $T = T_0 = 300\,K$

$N_i = (N/Z)\, g_i e^{-\varepsilon_i/k_B T} \Rightarrow \boxed{N_2/N_1 = 3e^{-1} = 1.104}$

and $\boxed{N_3/N_1 = 5e^{-2} = 0.677}$ at $T = T_0 = 300\,K$

(b) Average energy $\rightarrow \langle \varepsilon \rangle = (N_1/N)\varepsilon_1 + (N_2/N)\varepsilon_2 + (N_3/N)\varepsilon_3$ 6/

$= \frac{1}{Z}\left(g_1 \varepsilon_1 e^{-\varepsilon_1/k_B T} + g_2 \varepsilon_2 e^{-\varepsilon_2/k_B T}\right) = \frac{\varepsilon}{Z}\left[3e^{-1} + 2(5)e^{-2}\right]$

$\rightarrow \boxed{\langle \varepsilon \rangle = 0.884\,\varepsilon}$ at $T = T_0 = 300\,K$

(c) $N_3 = N_2 \Rightarrow g_3 e^{-\varepsilon_2/k_B T} = g_2 e^{-\varepsilon_2/k_B T}$ 6/

$\Rightarrow 5e^{-2\varepsilon/k_B T} = 3e^{-\varepsilon/k_B T} \Rightarrow e^{-\varepsilon/k_B T} = 3/5 \Rightarrow \varepsilon/k_B T = \ln(5/3)$

$\Rightarrow \boxed{T = \varepsilon/k_B \ln(5/3) = T_0/\ln(5/3) = 587\,K}$

(2) (a) $Z = aVT^4 \Rightarrow U = (N k_B T^2/Z)(\partial Z/\partial T)$ 4/

$\partial Z/\partial T = 4aVT^3 \Rightarrow \boxed{U = 4N k_B T}$

(b) For indistinguishable particles, 4/

$S = \frac{U}{T} + N k_B\left(1 - \ln\frac{N}{Z}\right) \rightarrow \boxed{S = N k_B\left[5 + \ln\left(\frac{aVT^4}{N}\right)\right]}$

(c) $P = N k_B T \frac{1}{Z}\frac{\partial Z}{\partial V} \rightarrow \boxed{P = \frac{N k_B T}{V}}$ 4/

(d) For indistinguishable particles, 4/

$\boxed{\mu = k_B T \ln\left(\frac{N}{Z}\right) = -k_B T \ln\left(\frac{aVT^4}{N}\right)}$

(e) $G = U - TS + PV = 4Nk_B T - 5Nk_B T - Nk_B T \ln\left(\frac{aVT^4}{N}\right) + Nk_B T$ 4/

$\rightarrow \underline{\underline{G = -Nk_B T \ln(aVT^4/N) = N\mu}}$

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(3) @ $Z = e^{-T_{vib}/2T} / (1 - e^{-T_{vib}/T})$ with $T_{vib}/T = 2$ 1/

$\rightarrow \boxed{Z = e^{-1}/(1-e^{-2}) = 0.425}$

$N_j = (N/Z) e^{-\varepsilon_j/k_B T} = (N/Z) e^{-(j+1/2)T_{vib}/T} = (N/Z) e^{-(2j+1)}$

$N = 1000 \rightarrow \boxed{N_0 = 865,\ N_1 = 117,\ N_2 = 16}$ for $j = 0, 1, 2$

(b) $U = N k_B T^2 \frac{\partial(\ln Z)}{\partial T} = N h f \left(\frac{1}{2} + \frac{1}{e^{hf/k_B T} - 1} \right) = N k_B T_{vib} \left(\frac{1}{2} + \frac{1}{e^2 - 1} \right)$ 5/

$\rightarrow \boxed{U = 657\ k_B T_{vib}}$

(c) For distinguishable particles, $S = U/T + N k_B \ln Z$

Now $U/T = 657\ k_B T_{vib}/T = 1313\ k_B$ 4/

and $N \ln Z = -855\ k_B \Rightarrow \boxed{S = 458\ k_B}$

(d) $F = U - TS = U - \frac{1}{2} T_{vib} S \rightarrow \boxed{F = 427\ k_B T_{vib}}$ 4/

(4) @ $Z = \sum_m e^{-\varepsilon_m/k_B T}$ with $\varepsilon_m = -m \mu_B B \Rightarrow \varepsilon_m/k_B T = -m\eta$

with $\eta = \mu_B B/k_B T \rightarrow Z = e^{3/2\eta} + e^{1/2\eta} + e^{-1/2\eta} + e^{-3/2\eta}$ 3×3

Now $e^x + e^{-x} = 2\cosh(x) \Rightarrow \boxed{Z = 2[\cosh(3/2\eta) + \cosh(1/2\eta)]}$

$B = 1.5\ T \Rightarrow \mu_B B = 8.683 \times 10^{-5}\ eV$

$T = 1.2\ K \Rightarrow k_B T = 1.034 \times 10^{-4}\ eV$ $\Rightarrow \eta = 0.8398$

$\rightarrow \boxed{Z = 5.99}$

(b) $N_m = (N/Z) e^{-\varepsilon_m/k_B T} = (N/Z) e^{m\eta}$ 5/

$\rightarrow \boxed{\begin{aligned} N_{-3/2} &= 0.047\,N & N_{-1/2} &= 0.110\,N \\ N_{1/2} &= 0.254\,N & N_{3/2} &= 0.589\,N \end{aligned}}$

(c) $\langle \varepsilon \rangle = \frac{1}{N} \sum_m \varepsilon_m N_m = \left(+\frac{3}{2} N_{-3/2} + \frac{1}{2} N_{-1/2} - \frac{1}{2} N_{1/2} - \frac{3}{2} N_{3/2} \right) \frac{\mu_B B}{N}$ 5/

$\rightarrow \boxed{\langle \varepsilon \rangle = -0.884\ \mu_B B = -7.68 \times 10^{-5}\ eV}$

(d) $\boxed{\langle \mu \rangle = \left(-\frac{3}{2} N_{-3/2} - \frac{1}{2} N_{-1/2} + \frac{1}{2} N_{1/2} + \frac{3}{2} N_{3/2} \right) \frac{\mu_B}{N} = 0.884\ \mu_B}$ 4/