PHY 3513
Test IV — 5 Dec 2016

— Solutions —

1 (a) $M = 63.5\times10^{-3}$ kg/mole, $\rho = 8.9\times10^{3}$ kg/m^3

7/ $\Rightarrow N/V = (\rho/M)N_A = 8.44\times10^{28}\ \text{m}^{-3}$

$h^2/2m_e = 2.410\times10^{-37}\ \text{J-m}^2 \rightarrow \varepsilon_F = (h^2/2m_e)(3N/8\pi V)^{2/3}$

$\Rightarrow \boxed{\varepsilon_F = 1.124\times10^{-18}\ \text{J} = 7.017\ \text{eV}}$

$\boxed{T_F = \varepsilon_F/k_B = 8.14\times10^{4}\ \text{K}}$ and $\boxed{P = \tfrac{2}{5}(N/V)\varepsilon_F = 3.79\times10^{10}\ \text{Pa}}$

(b) $\mu \cong \varepsilon_F\left[1 - \pi^2/12\,(T/T_F)^2\right]$

4/ so $\mu = 0.99\,\varepsilon_F \Rightarrow \pi^2/12\,(T/T_F)^2 = 0.01 \Rightarrow T = (\sqrt{0.12}/\pi)T_F$

$\rightarrow \boxed{T = 0.1103\,T_F = 8980\ \text{K}}$

4/ (c) $f(\varepsilon) = \left[e^{(\varepsilon-\mu)/k_BT} + 1\right]^{-1}$ with $\varepsilon = 10\,k_BT = 7.738$ eV

$\mu = 0.99\,\varepsilon_F = 6.947\ \text{eV} \Rightarrow (\varepsilon-\mu)/k_BT = 1.022$

$\Rightarrow \boxed{f(\varepsilon) = 0.265}$

2 (a) Wien's displ. law $\rightarrow \lambda_{max}T = 2.90\times10^{-3}$ m-K

8/ $\lambda_{max} = 4.80\times10^{-7}\ \text{m} \Rightarrow \boxed{T = 6040\ \text{K}}$

Stefan-Boltzmann law $\rightarrow \mathcal{J} = \sigma T^4$

$\sigma = \tfrac{2}{15}(\pi^5 k_B^4/h^3c^2) = 5.67\times10^{-8}\ \text{W/m}^2\text{-K}^4 \Rightarrow \mathcal{J} = 7.553\times10^{7}\ \frac{\text{W}}{\text{m}^2}$

$R = 6.96\times10^{8}\ \text{m} \rightarrow$ area $A = 4\pi R^2 = 6.087\times10^{18}\ \text{m}^2$

$\Rightarrow \boxed{P = \mathcal{J}A = 4.60\times10^{26}\ \text{W}}$

(b) $U = \left[8\pi^5k_B^4/15(hc)^3\right]T^4V = (1.007\ \text{J/m}^3)\,V$

7/ $V = \tfrac{4}{3}\pi R^3 = 1.412\times10^{27}\ \text{m}^3 \Rightarrow \boxed{U = 1.42\times10^{27}\ \text{J}}$

$\langle\varepsilon\rangle = U/N = (U/V)/(N/V)$

$N/V = (2.404)\,8\pi(k_BT/hc)^3$

$k_BT/hc = 4.197\times10^{5}\ \text{m}^{-1} \Rightarrow N/V = 4.468\times10^{18}\ \text{m}^{-3}$

$\Rightarrow \boxed{\langle\varepsilon\rangle = 2.254\times10^{-19}\ \text{J} = 1.41\ \text{eV}}$

Alternatively, $(U/V)/(N/V) = \left[\pi^4/(2.404)15\right]k_BT = 2.701\,k_BT$

$k_BT = 0.5204\ \text{eV} \rightarrow \langle\varepsilon\rangle = 1.41\ \text{eV}$

3 a $\varepsilon = \frac{1}{2} m v^2 \Rightarrow v = \sqrt{2\varepsilon/m} \Rightarrow \langle v \rangle = \frac{1}{N} \int_0^\infty \sqrt{2\varepsilon/m}\; N(\varepsilon)\, d\varepsilon$

8/ At $T=0$, this reduces to

$$\langle v \rangle = \frac{1}{N}\sqrt{\frac{2}{m}} \int_0^{\varepsilon_F} \varepsilon^{1/2} g(\varepsilon)\, d\varepsilon \quad \rightarrow \quad \frac{g(\varepsilon)}{N} = \frac{3}{2}\frac{\varepsilon^{1/2}}{\varepsilon_F^{3/2}}$$

$$\Rightarrow \langle v \rangle = \tfrac{3}{2}\sqrt{2/m}\; \varepsilon_F^{-3/2} \int_0^{\varepsilon_F} \varepsilon\, d\varepsilon = \tfrac{3}{4}\sqrt{2\varepsilon_F/m}$$

$$\varepsilon_F = \tfrac{1}{2} m v_F^2 \Rightarrow \sqrt{2\varepsilon_F/m} = v_F \Rightarrow \boxed{\langle v \rangle = \tfrac{3}{4} v_F}$$

b $U \propto V \Rightarrow \partial U/\partial V = U/V = u$ and $P = \frac{1}{3} u$

7/ $$\Rightarrow \frac{\partial U}{\partial V} = T\frac{\partial P}{\partial T} - P \Rightarrow u = \frac{1}{3}\left(T\frac{du}{dT} - u\right) \Rightarrow \underline{T\frac{du}{dT} = 4u}$$

$$\Rightarrow du/u = 4(dT/T) \Rightarrow \boxed{u = AT^4}$$

4 Let $x = (\varepsilon - \mu)/k_B T \Rightarrow f_{BE} = (e^x - 1)^{-1}$, $f_{FD} = (e^x + 1)^{-1}$ and $f_{MB} = e^{-x}$

$$\Rightarrow f_{BE} - f_{FD} = \frac{1}{e^x - 1} - \frac{1}{e^x + 1} = \frac{2}{e^{2x} - 1}$$

so $f_{BE} - f_{FD} = f_{MB}/1000 \Rightarrow 2/(e^{2x} - 1) = e^{-x}/1000$

$$\Rightarrow \frac{e^x - e^{-x}}{2} = 1000 \Rightarrow \sinh(x) = 1000 \Rightarrow \underline{x = 7.601}$$

$T = 300\text{ K} \Rightarrow k_B T = 0.02585\text{ eV}$

$$\Rightarrow \boxed{\varepsilon - \mu = (k_B T)x = 0.196\text{ eV}}$$

Alternatively, since x is large, $\sinh(x) \cong e^x/2$

$$\Rightarrow x \cong \ln(2000) = 7.601$$

or solve the quadratic equation $e^x - e^{-x} = 2000$

$$\Rightarrow (e^x)^2 - 2000\, e^x - 1 = 0$$