PHZ 3113 – PROBLEM SET 4

1) Prove the following statements:

a) If the *same* linear transformation diagonalizes two matrices A and B, then the matrices A and B must commute. *Hint*: use the fact that the *diagonalized* matrices must commute since they are both diagonal.

b) The eigenvalues of an arbitrary 2 X 2 matrix A satisfy the equation

$$\lambda^2 - \lambda \, \mathrm{Tr}(A) + \det(A) = 0$$

2) For each of the 3 X 3 matrices below, find the eigenvalues of the matrix and a set of eigenvectors that are *normalized* and *orthogonal* to one another (an orthonormal set).

a) 1 1 0 (row 1), 1 0 1 (row 2), 0 1 1 (row 3)

b) 5 0 sqrt(3) (row 1), 0 3 0 (row 2), sqrt(3) 0 3 (row 3)

c) 1 -1 -1 (row 1), -1 1 -1 (row 2), -1 -1 1 (row 3)

3) **Text Problem 1.23 –** you will first need to find the eigenvalues and the equations satisfied by the corresponding eigenvectors for each of the two matrices. Then find a set of eigenvectors that satisfies both sets of equations. Note that you should be able to make the eigenvectors orthogonal to each other and normalized as well.

4) **Text Problem 1.27 –** first determine the eigenvalues of the matrix H and the corresponding set of orthonormal eigenvectors. The transformation matrix U has these eigenvectors as its columns. Verify that the transformation indeed diagonalizes the matrix H.

5) The vectors **f** and **g** satisfy the equations A**f**=λ**g** and A^T**g**=λ**f**, where A is a matrix with *real* elements and λ is a scalar.

a) Prove that **f** is an eigenvector of the matrix A^TA with eigenvalue λ^2.

b) Prove that **g** is an eigenvector of the matrix AA^T with eigenvalue λ^2.

c) Let A be the 2 X 2 matrix with rows 2/a 2/a (row 1) and 1/a -4/a (row 2) where a=sqrt(5). For this matrix A, find the two possible values of λ^2 defined above and the corresponding *normalized* vectors **f** and **g**.

6) **Text Problem 1.31 –** note in this problem, that the quadratic quantity Q=x·A·x (or x^TAx in the text's notation), where A is a *symmetric* matrix. You will need to first find the matrix A and its normalized eigenvectors. The signs of the corresponding eigenvalues determine the type of conic section that the equation Q=110 represents. For elliptical conic sections, the eigenvectors are along the directions of the major and minor axes of the ellipse in the original coordinate system.

7) **Text Problem 1.32 –** this problem is essentially a three dimensional version of the previous problem. Note that the longest axis of the ellipsoid corresponds to the *smallest* of the matrix's eigenvalues.

8) **Text Problem 1.33 –** this problem is similar to the two previous ones, except that one of the matrix's eigenvalues is two-fold degenerate (two eigenvectors with the same eigenvalue). This means that the ellipsoid has a symmetry axis, which is in the direction of the eigenvector corresponding to the *non-degenerate* eigenvalue.