PHZ 3113 – PROBLEM SET 6

1) The function f(t) is defined by $f(t)=\exp(-at)$ for $t \geq 0$.

a) Derive the Fourier *sine* and Fourier *cosine* transformations of this function.

b) Using the results of part a, show that the function exp(-at) can be represented by either of the integrals below, where the integration ranges are from 0 to ∞.

$$\exp(-at) = (2/\pi) \int \omega \sin(\omega t)/(a^2+\omega^2)\, d\omega$$

$$\exp(-at) = (2a/\pi) \int \cos(\omega t)/(a^2+\omega^2)\, d\omega$$

2) **Text Problem 5.2 –** in part a, note that if you make a change of variable in the Fourier transform integral from x to x'=x+a, the function f(x) will not be affected due to the periodicity assumption. The result in part (c) can also be easily derived by making a change of variable.

3) Consider the differential equation, $-D\, d^2f/dx^2 + K^2 D\, f(x) = Q\delta(x)$, where K, D, and Q are constants, and $\delta(x)$ is the Dirac δ-function.

a) Derive the equation satisfied by the Fourier transform of f(x) and solve it.

b) Perform the inverse Fourier transform to obtain the function f(x) that satisfies the original differential equation. Note that since the Fourier transform is symmetric, the relevant transform is the cosine one. For $x>0$, the required integral is given in problem 1. For $x<0$, just replace x by -x.

4) **Text Problem 5.1 –** note that the function is *symmetric*, so that a cosine Fourier transformation is appropriate.

5) The function $q(x,t)$ satisfies the differential equation $\partial^2 q/\partial x^2 = \partial q/\partial t$. Define the function $Q(k,t)$ to be the Fourier transform of q with respect to x.

a) Derive the differential equation satisfied by Q and show that its solution has the form $Q(k,t) = A \exp(-k^2 t)$, where A is a constant.

b) Suppose that at time t=0, the function q is given by $q(x,t=0)=S\,\delta(x)$, where δ is the Dirac δ-function, and S is a constant. Show that this condition requires that the constant A be equal to $S/\mathrm{sqrt}(2\pi)$.

c) By evaluating the inverse Fourier transform of Q, find the function $q(x,t)$ that satisfies the original differential equation with the initial condition given in part b. To carry out the required integral, define a new variable u such that $u^2=k^2t-ikx-x^2/4t$ and make use of the result $\int \exp(-u^2)\,du = \mathrm{sqrt}(\pi)$, where the integral is from $-\infty$ to $+\infty$.

6) **Text Problem 5.7** – note that the function is *symmetric*, so that a cosine Fourier transformation is appropriate. In evaluating the convolution integral, you will have to determine the region over which the product integrand is non-zero.

7) **Text Problem 5.14** – note that the result can be obtained using Parseval's theorem.

8) Consider the function f(x) defined by the relations

$$f(x) = 1-|x|/2 \qquad \text{for } -2\le x\le 2$$
$$= 0 \qquad \text{otherwise}$$

a) Noting that the function is *symmetric*, derive the Fourier cosine transform of the function.

b) Using the result of part a and Parseval's theorem for Fourier transforms, show that

$$\int (\sin k/k)^4\, dk = 2\pi/3$$

where the integral ranges from $-\infty$ to $+\infty$.

9) The three dimensional Fourier transform of the function f(**r**) is given by

$$F(\mathbf{k})=1/[(2\pi)^{3.2}k^2]$$

where k is the magnitude of the vector **k**. Show that f(**r**) = 1/(4πr), where r is the magnitude of the position vector **r**. The required three dimensional integral can be most easily evaluated using spherical coordinates with the k-space z-axis chosen in the direction of the position vector **r**. To perform the k-space radial integral, you will need the result

$$\int \sin u/u \, du = \pi/2$$

where the integral ranges from 0 to ∞.