PHZ 3113 – PROBLEM SET 7

1) In polar form, the complex quantity z=x+*i*y is given by $z=re^{i\theta}$.

a) Express 1/z in terms of x and y and in polar form.

b) Suppose z=±*i*. Find the real and imaginary parts of the square root of z. *Hint*: express z in polar form before taking the square root. Note that together, ±*i* have *four* square roots.

2) The sine and cosine functions of a complex argument z=x+*i*y can be expressed as

$$\sin(z) = -\tfrac{1}{2}\, i\, [\exp(iz) - \exp(-iz)], \quad \cos(z) = \tfrac{1}{2}\, [\exp(iz) + \exp(-iz)]$$

a) Using these expressions, show that

$$\sin(z) = \sin x \cosh y + i \cos x \sinh y$$
$$\cos(z) = \cos x \cosh y - i \sin x \sinh y$$

where sinh y and cosh y are the hyperbolic sine and cosine functions.

b) Use the relations of part a to show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$
$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

c) Use the relations of part a to determine the values of x and y for which sin(z)=0 and for which cos(z)=0. Note that for a complex quantity to be zero, the real and imaginary parts must be zero separately.

3) Derive the following identities for the inverse sine and tangent functions. *Hint*: first express the sine and tangent functions in terms of exponentials, then solve for the exponentials.

a) $\sin^{-1}(z) = -i \ln[iz \pm \mathrm{sqrt}(1-z^2)]$

b) $\tan^{-1}(z) = \tfrac{1}{2}\, i \ln[i+z)/(i-z)]$

4) Prove the following properties of an *analytic* function f(z) = u(x,y) + *i*v(x,y).

a) If f(z) and its complex conjugate are *both* analytic, then the function must be constant, i.e, u and v must both be constants. *Hint*: the Cauchy-Riemann relations must be satisfied by both the original function and the complex conjugate function if both are analytic. Show that this gives a contradiction if u and v are not both constant.

b) The function f*(z*) is analytic if f(z) is (* indicates complex conjugate).

5) **Text Problem 14.1 –** note that both Cauchy-Riemann relations have to be satisfied by the function. You should be able to express the complete function (both real and imaginary parts) as a single function of the complex argument z.

6) Consider the integral of a complex function from the point x=0, y=0 in the complex plane to the point x=1, y=1. Consider two different paths: (i) (0,0) --> (1,0) --> (1,1) and (ii) (0,0) --> (0,1) --> (1,1).

a) Show that if f(z)=z, which is an analytic function, the two paths yield the same result. Note that when integrating in the y-direction, dz=*i*dy.

b) Show that if f(z)=z* (complex conjugate of z), which is *not* an analytic function (how would you prove this?), the two paths yield *different* results.

7) **Text Problem 14.7 –** in the text section 14.7, it was shown, using the Cauchy-Riemann relations, that the real and imaginary parts of *any* analytic function are *separate* solutions of Laplace's equation. Since Laplace's equation is linear and homogeneous, any linear combination of solutions is another solution. What is desired is a linear combination of solutions that satisfies the given boundary conditions, i.e., which gives the specified values on the boundaries of the square region in the xy-plane.

8) Consider the function f(z) = 1/[exp(z) – 1]. Show that up to terms of order z, the Laurent expansion of this function is given by f(z) ≅ 1/z - 1/2 + z/12. *Hint*: begin with the Taylor series for exp(z) and then combine this with the binomial expansion for $(u+1)^{-1}$, with u suitably chosen. Note that you have to make sure that you collect together *all* the terms of each order in z.

9) Consider the two mappings given by $w = z + (1/z)$ and $w = z - (1/z)$

a) For each of these two mappings, show that a circle of radius R in the z-plane maps into an ellipse in the w-plane. In each case, determine the semi-major and semi-minor axis lengths of the ellipse and make a sketch that shows how the ellipse is oriented.

b) Determine what happens to the ellipses in part a when R --> 1.

10) Consider the transformation from the z-plane to the w-plane that is defined by the relation $e^z = (a-w)/(a+w)$, where a is a constant.

a) Show that the transformation into w-space of the real axis in z-space is given by

$$w(x,y=0) = -a \tanh(x/2)$$

b) Show that the transformation into w-space of the imaginary axis in z-space is given by

$$w(x=0,y) = -ia \tan(y/2)$$