**PHZ 3113 – PROBLEM SET 8**

1) Using Cauchy's integral theorems, including the extension of the second theorem to derivatives of arbitrary order, show that

$$\oint (z-z_o)^n\, dz = 0 \text{ if } n\neq -1, \qquad \oint (z-z_o)^{-1}\, dz = 2\pi i$$

where in both cases, the integral is around a closed contour in the complex plane that surrounds the point $z=z_o$.

2) **Text Problem 15.6 –** note that since the cosine function is symmetric under reflection about the real axis, the integral of any function of the type f(cos $\theta$) over the semi-circle from 0 to $\pi$ is just half the integral over the full circle from 0 to $2\pi$. Note also that only one of the two poles of f(z) lies within the unit circle.

3) Let $f(x) = \cos(kx)/(x^2+a^2)$ with $a>0$. Extend the function into the complex plane by replacing cos x by exp($ikz$). Then by making use of a semicircular contour like that of Fig. 15.5 in the text, show that

$$\int f(x)\, dx = (\pi/a)\exp(-ka)$$

where the integral ranges from $-\infty$ to $+\infty$. Note that f(z) has two simple poles, but only one of them lies within the integration contour.

4) **Text Problem 15.8 –** note that the integrand is symmetric so the integral from 0 to $\infty$ is just half the integral from $-\infty$ to $\infty$. Extend the function into the complex plane by replacing sin $\alpha$t by $-i\exp(i\alpha z)$ and make use of the same contour as in the previous problem.

5) Let $f(x) = (x^2+a^2)^{-2}$ with $a>0$. Extend f(x) into the complex plane and then by making use of the same contour as in the previous two problems and noting that f(x) is symmetric in x, show that

$$\int f(x)\, dx = \pi/(4a^3)$$

where the integral is from 0 to $\infty$. Note that the poles of f(z) are *double*, not single poles.

6) Let $f(\theta) = 1/(a+\cos\theta)^2$, with $a>1$. By transforming the integral over $\theta$ to an integral over $z=\exp(i\theta)$ around a unit circle in the complex plane, show that

$$\int f(\theta)\, d\theta = \pi a/(a^2-1)^{3/2}$$

where the integral is from 0 to $\pi$. Note that the cosine function is symmetric under reflection about the real axis, so that the integral of any function of the type $f(\cos\theta)$ between 0 and $\pi$ is just half the integral from 0 to $2\pi$. Note also, that the poles of the function $f(z)$ in the complex plane are *double* poles, not single poles.

7) Let $f(x) = [\cos(bx)-\cos(ax)]/x^2$ with $a>b>0$. Extend this function into the complex plane by replacing $\cos(ax)$ and $\cos(bx)$ by $\exp(iaz)$ and $\exp(ibz)$. Note that $f(z)$ has a double pole on the real axis at $z=0$. To avoid this pole, make use of a contour similar to that shown in Fig !5.6 of the text but with the small semicircle at $z=0$. Note that, as discussed in lecture, the integral over this small semicircle is just equal to $-\pi i$ times the residue of the double pole at $z=0$. By evaluating the integral of $f(z)$ over the specified contour, show that

$$\int f(x)\, dx = \pi(a-b)$$

where the integral is from $-\infty$ to $+\infty$.

8) **Text Problem 15.11 –** note that since the function given has a branch point at the origin ($\alpha<1$), a branch cut is required. Make use of the contour shown in Fig. 15.7 of the text. You will need to demonstrate that the integrals over both the circles vanish and that the integrands on the upper and lower straight sections just differ by a phase factor equal to $\exp(-2\pi\alpha i)$, which is not equal to one since $\alpha$ is not an integer.

9) Let $f(x) = (1+x^4)^{-1}$. Extend this function into the complex plane and then integrate it around a contour consisting of the positive real axis, the positive imaginary axis, and a quarter circle of radius $R \to \infty$ joining the two. Note that the integrand along the imaginary axis is related to that along the real axis by a simple phase factor and that only one of the four poles of $f(z)$ is within the contour. Thereby demonstrate that

$$\int f(x)\, dx = \pi/2^{3/2}$$

where the integral is from 0 to $\infty$.