Set 1

— Solutions —

(1) (a) $\vec{r} = r\cos(\omega t)\,\hat{x} + r\sin(\omega t)\,\hat{y}$

$\Rightarrow d\vec{r}/dt = \omega r[-\sin(\omega t)\hat{x} + \cos(\omega t)\hat{y}] + (dr/dt)[\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}]$

Now the second term $= 1/r\,(dr/dt)\,\vec{r}$ and $\vec{r}\times\vec{r} = 0$

Also $[\cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}] \times [-\sin(\omega t)\hat{x} + \cos(\omega t)\hat{y}]$

$= -\sin^2\omega t(\hat{y}\times\hat{x}) + \cos^2(\omega t)(\hat{x}\times\hat{y}) = [\sin^2(\omega t) + \cos^2\omega t]\,\hat{z} = \hat{z}$

$\Rightarrow \boxed{\vec{r}\times d\vec{r}/dt = \omega r^2\hat{z}}$

(b) If $dr/dt = 0$, then $d\vec{r}/dt = \omega r[-\sin(\omega t)\hat{x} + \cos(\omega t)\hat{y}]$

$\Rightarrow \boxed{d^2\vec{r}/dt^2 = \omega^2 r[-\cos(\omega t)\hat{x} - \sin(\omega t)\hat{y}] = -\omega^2\vec{r}}$

[Text 2.9] (2) (a) $\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = \partial[(\vec{a}\times\vec{b})_x]/\partial x + \partial[(\vec{a}\times\vec{b})_y]/\partial y + \partial[(\vec{a}\times\vec{b})_z]/\partial z$

$= \frac{\partial}{\partial x}(a_y b_z - a_z b_y) + \frac{\partial}{\partial y}(a_z b_x - a_x b_z) + \frac{\partial}{\partial z}(a_x b_y - a_y b_x)$

$= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)b_z + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)b_y + \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)b_x$

$+ \left(\frac{\partial b_z}{\partial x} - \frac{\partial b_x}{\partial z}\right)a_y + \left(\frac{\partial b_x}{\partial y} - \frac{\partial b_y}{\partial x}\right)a_z + \left(\frac{\partial b_y}{\partial z} - \frac{\partial b_z}{\partial y}\right)a_x$

$= (\vec{\nabla}\times\vec{a})_z b_z + (\vec{\nabla}\times\vec{a})_y b_y + (\vec{\nabla}\times\vec{a})_z b_z - (\vec{\nabla}\times\vec{b})_y a_y - (\vec{\nabla}\times\vec{b})_z a_z - (\vec{\nabla}\times\vec{b})_x a_x$

$= (\vec{\nabla}\times\vec{a})\cdot\vec{b} - (\vec{\nabla}\times\vec{b})\cdot\vec{a} \Rightarrow \boxed{\vec{\nabla}\cdot(\vec{a}\times\vec{b}) = (\vec{\nabla}\times\vec{a})\cdot\vec{b} - (\vec{\nabla}\times\vec{b})\cdot\vec{a}}$

[Text 2.12] (3) $\vec{a} = x^2y\,\hat{x} + xyz\,\hat{y} + z^2y\,\hat{z}$

$\Rightarrow \vec{\nabla}\times\vec{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{x} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{y} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{z}$

$= (z^2 - xy)\,\hat{x} + (yz - x^2)\,\hat{z}$

$\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{a}) = \frac{\partial[(\vec{\nabla}\times\vec{a})_z]}{\partial y}\hat{x} + \left(\frac{\partial[(\vec{\nabla}\times\vec{a})_x]}{\partial z} - \frac{\partial[(\vec{\nabla}\times\vec{a})_z]}{\partial x}\right)\hat{y} - \frac{\partial[(\vec{\nabla}\times\vec{a})_x]}{\partial y}\hat{z}$

(3) (continued) $\Rightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{a}) = z\,\hat{x} + 2(z+x)\,\hat{y} + x\,\hat{z}$

On the other side of the equation,

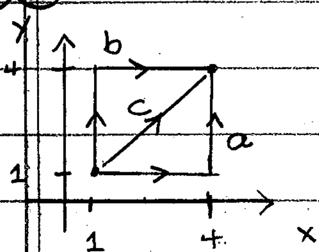
$$\vec{\nabla}\cdot\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} = 2xy + xz + 2zy$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) = (2y+z)\,\hat{x} + 2(x+z)\,\hat{y} + (2y+x)\,\hat{z}$$

$$\nabla^2\vec{a} = (\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)\,\vec{a} = 2y\,\hat{x} + 2y\,\hat{z}$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \nabla^2\vec{a} = z\,\hat{x} + 2(x+z)\,\hat{y} + x\,\hat{z}$$ $\rightarrow$ agree

(4) (a) $\vec{F} = -kx\,\hat{x} - ky\,\hat{y}$ and $W = -\int \vec{F}\cdot d\vec{r}$



$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy \Rightarrow -\vec{F}\cdot d\vec{r} = k(x\,dx + y\,dy)$

path a $\rightarrow W = k\int_1^4 x\,dx + k\int_1^4 y\,dy$

$= \tfrac{1}{2}kx^2\Big|_1^4 + \tfrac{1}{2}ky^2\Big|_1^4 \rightarrow \boxed{W = 15k}$

(b) path b $\rightarrow$ same thing with x/y integrals interchanged

(c) path c $\rightarrow$ now $y = x \Rightarrow dy = dx \Rightarrow y\,dy = x\,dx$

$\Rightarrow W = k\int(x\,dx + y\,dy) = 2k\int_1^4 x\,dx \rightarrow \boxed{W = 15k}$

The elastic force is conservative, so work against force does not depend on the path

[Text 3.2] (5) $\vec{Q} = [3x^2(y+z) + y^3 + z^3]\,\hat{x} + [3y^2(x+z) + x^3 + z^3]\,\hat{y} + [3z^2(x+y) + x^3 + y^3]\,\hat{z}$

$$\vec{\nabla}\times\vec{Q} = \left(\frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z}\right)\hat{x} + \left(\frac{\partial Q_x}{\partial z} - \frac{\partial Q_z}{\partial x}\right)\hat{y} + \left(\frac{\partial Q_y}{\partial x} - \frac{\partial Q_x}{\partial y}\right)\hat{z}$$

$$= [3(z^2+y^2) - 3(y^2-z^2)]\,\hat{x} + [3(x^2+z^2) - 3(z^2+x^2)]\,\hat{y} + [3(y^2+x^2) - 3(x^2+y^2)]\,\hat{z}$$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{Q} = 0} \Rightarrow$ Q conservative

(5) (continued) $\Rightarrow \vec{Q} = \vec{\nabla}\phi$

$$\frac{\partial\phi}{\partial x} = Q_x = 3x^2(y+z)+y^3+z^3 \Rightarrow \phi = x^3(y+z)+x(y^3+z^3)+f(y,z)$$

now $\partial\phi/\partial y = Q_y = 3y^2(x+z)+x^3+z^3$

$\Rightarrow x^3+3y^2x+\partial f/\partial y = 3y^2(x+z)+x^3+z^3 \Rightarrow \partial f/\partial y = 3y^2z+z^3$

$\Rightarrow f(y,z) = y^3z+z^3y+g(z) \Rightarrow \phi = x^3(y+z)+y^3(x+z)+z^3(x+y)+g(z)$

$\partial\phi/\partial z = Q_z = 3z^2(x+y)+x^3+y^3$

$\Rightarrow x^3+y^3+3z^2(x+y)+\partial g/\partial z = 3z^2(x+y)+x^3+y^3 \Rightarrow \partial g/\partial z = 0$

$\Rightarrow g = c \rightarrow \boxed{\phi = x^3(y+z)+y^3(x+z)+z^3(x+y)+c}$

$$\int_A^B \vec{Q}\cdot d\vec{r} = \int_A^B \vec{\nabla}\phi\cdot d\vec{r} = \int_A^B d\phi = \phi_B-\phi_A$$

$A=(1,-1,1) \rightarrow \phi_A = -2$ and $B=(2,1,2) \rightarrow \phi_B = 2(24)+4 = 52$

$\rightarrow \boxed{\int_A^B \vec{Q}\cdot d\vec{r} = 54}$

[Text 3.3] (6) $\vec{F} = xy^2\hat{x}+2\hat{y}+x\hat{z}$ for $x=ct$, $y=c/t$, $z=d$, $1\le t\le 2$

(a) $\int_1^2 \vec{F}\,dt = \int_1^2\left[\frac{c^3}{t}\hat{x}+2\hat{y}+ct\hat{z}\right]dt = (c^3\ln 2)\hat{x}+2\hat{y}+\frac{3}{2}c\,\hat{z}$

(b) $y = c/t \Rightarrow dy = -c/t^2\,dt$

$\Rightarrow \int_1^2 \vec{F}\,dy = -c\int_1^2 \vec{F}\frac{dt}{t^2} = -c\int_1^2\left(\frac{c^3}{t^3}+\frac{2}{t^2}+\frac{c}{t}\right)dt$

$\Rightarrow \boxed{\int_1^2 \vec{F}\,dy = -\left[\frac{3}{8}c^4+c+c^2\ln 2\right]}$

(c) $x = ct \Rightarrow dx = c\,dt \Rightarrow \vec{F}\cdot d\vec{r} = F_x dx+F_y dy = c(F_x-F_y/t^2)dt$

$\Rightarrow \boxed{\int_1^2 \vec{F}\cdot d\vec{r} = c\int_1^2\left(\frac{c^3}{t}-\frac{2}{t^2}\right)dt = c^4\ln 2 - c}$

(7) Define a vector field $\vec{A} \equiv u\vec{\nabla}v + v\vec{\nabla}u$

Stokes thm $\Rightarrow \oint \vec{A}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, dS$ for any closed loop

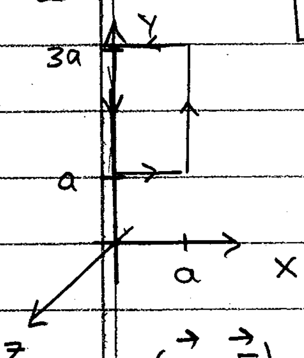
But $\vec{\nabla}\times\vec{A} = \vec{\nabla}\times[(u\vec{\nabla}v) + v\vec{\nabla}u]$

$= \vec{\nabla}u\times\vec{\nabla}v + u\vec{\nabla}\times\vec{\nabla}v + \vec{\nabla}v\times\vec{\nabla}u + v\vec{\nabla}\times\vec{\nabla}u = 0$

(since $\vec{\nabla}v\times\vec{\nabla}u = -\vec{\nabla}u\times\vec{\nabla}v$ and $\vec{\nabla}\times\vec{\nabla}u = \vec{\nabla}\times\vec{\nabla}v = 0$)

$\Rightarrow \vec{\nabla}\times\vec{A} = 0 \Rightarrow \underline{\underline{\oint \vec{A}\cdot d\vec{r} = \oint (u\vec{\nabla}v + v\vec{\nabla}u)\cdot d\vec{r} = 0}}$

Text 3.28 (8) $\vec{F} = F_0\left[\left(\frac{y^3}{3a^3} + \frac{y}{a}e^{xy/a^2} + 1\right)\hat{x} + \left(\frac{xy^2}{a^3} + \frac{x+y}{a}e^{xy/a^2}\right)\hat{y} + \frac{z}{a}e^{xy/a^2}\hat{z}\right]$



counterclockwise loop in xy plane

$\Rightarrow \hat{n} = z$

$\Rightarrow \oint \vec{F}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{F})\cdot\hat{z}\, dS$

$(\vec{\nabla}\times\vec{F})\cdot\hat{z} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = F_0\left[\frac{y^2}{a^3} + \frac{1}{a}\left[1 + \frac{y}{a^2}(x+y)\right]e^{xy/a^2} - \frac{y^2}{a^3} - \frac{1}{a}\left(1 + \frac{xy}{a^2}\right)e^{\frac{xy}{a^2}}\right]$

$= (y^2/a^3)\, e^{xy/a^2} F_0$

$\Rightarrow \oint \vec{F}\cdot d\vec{r} = F_0\int_a^{3a} dy\, \frac{y^2}{a^3}\int_0^a dx\, e^{xy/a^2} = F_0\int_a^{3a} \frac{y}{a}(e^{y/a} - 1)\, dy$

$(u = \frac{y}{a}) \quad = F_0 a\int_1^3 u(e^u - 1)\, du = F_0 a\left[ue^u\Big|_1^3 - \int_1^3 e^u du - \frac{1}{2}u^2\Big|_1^3\right]$

$= F_0 a(3e^3 - e - e^3 + e - \frac{1}{2}8) \Rightarrow \boxed{\oint \vec{F}\cdot d\vec{r} = F_0 a(2e^3 - 4)}$