Set 2 — Solutions —

(1) (a) $\vec{r} = \rho\hat{\rho} + z\hat{z} \Rightarrow d\vec{r}/dt = (d\rho/dt)\,\hat{\rho} + \rho(d\hat{\rho}/dt) + (dz/dt)\,\hat{z}$

$\hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi \Rightarrow d\hat{\rho}/dt = (-\hat{x}\sin\phi + \hat{y}\cos\phi)(d\phi/dt)$

$-\hat{x}\sin\phi + \hat{y}\cos\phi = \hat{\phi} \Rightarrow \boxed{\vec{v} = (d\rho/dt)\,\hat{\rho} + \rho(d\phi/dt)\,\hat{\phi} + (dz/dt)\,\hat{z}}$

(b) $\vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\rho}{dt^2}\hat{\rho} + \dfrac{d\rho}{dt}\dfrac{d\hat{\rho}}{dt} + \dfrac{d\rho}{dt}\dfrac{d\phi}{dt}\hat{\phi} + \rho\dfrac{d^2\phi}{dt^2}\hat{\phi} + \rho\dfrac{d\phi}{dt}\dfrac{d\hat{\phi}}{dt} + \dfrac{d^2z}{dt^2}\hat{z}$

$\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \Rightarrow d\hat{\phi}/dt = -\hat{\rho}\,(d\phi/dt)$

$$\Rightarrow \boxed{\vec{a} = \left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2\right]\hat{\rho} + \left[\rho\frac{d^2\phi}{dt^2} + 2\left(\frac{d\rho}{dt}\right)\left(\frac{d\phi}{dt}\right)\right]\hat{\phi} + \frac{d^2z}{dt^2}\hat{z}}$$

(2) $\phi(\vec{r}) = (\vec{p}\cdot\vec{r})/(4\pi\varepsilon_0 r^3)$

$$\Rightarrow \vec{E} = -\vec{\nabla}\phi = \frac{-\vec{\nabla}(\vec{p}\cdot\vec{r})}{4\pi\varepsilon_0 r^3} - \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{1}{r^3}\right)$$

Can show that (constant $\vec{p}$) $\vec{\nabla}(\vec{p}\cdot\vec{r}) = \vec{p}$

$$\vec{\nabla}\left(\frac{1}{r^3}\right) = \frac{d}{dr}\left(\frac{1}{r^3}\right)\hat{r} = -\frac{3}{r^4}\hat{r} \Rightarrow \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0}\vec{\nabla}\left(\frac{1}{r^3}\right) = -\frac{3(\vec{p}\cdot\hat{r})\hat{r}}{4\pi\varepsilon_0 r^3}$$

$$\Rightarrow \boxed{\vec{E} = \frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{4\pi\varepsilon_0 r^3}}$$

(3) (a) Cartesian coordinates $\rightarrow \vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$

$\Rightarrow \vec{A}\cdot\vec{\nabla} = A_x\,\partial/\partial x + A_y\,\partial/\partial y + A_z\,\partial/\partial z$

$\partial\vec{r}/\partial x = \partial/\partial x\,(x\hat{x} + y\hat{y} + z\hat{z}) = \hat{x}$

Similarly, $\partial\vec{r}/\partial y = \hat{y}$, $\partial\vec{r}/\partial z = \hat{z}$

$\Rightarrow (\vec{A}\cdot\vec{\nabla})\vec{r} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z} \Rightarrow \boxed{(\vec{A}\cdot\vec{\nabla})\vec{r} = \vec{A}}$

(3) (b) spherical coords → $\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$

$\Rightarrow \vec{A}\cdot\vec{\nabla} = A_r\,\partial/\partial r + (A_\theta/r)\,\partial/\partial\theta + A_\phi/(r\sin\theta)\;\partial/\partial\phi$

$\vec{r} = r\sin\theta(\hat{x}\cos\phi + \hat{y}\sin\phi) + r\cos\theta\,\hat{z}$

$\Rightarrow \partial\vec{r}/\partial r = \vec{r}/r = \hat{r}$

$\partial\vec{r}/\partial\theta = r\cos\theta(\hat{x}\cos\phi + y\sin\phi) - r\sin\theta\,\hat{z} = r\hat{\theta}$

$\partial\vec{r}/\partial\phi = r\sin\theta(-\hat{x}\sin\phi + \hat{y}\cos\phi) = r\sin\theta\,\hat{\phi}$

$\Rightarrow(\vec{A}\cdot\vec{\nabla})\vec{r} = A_r\hat{r} + (A_\theta/r)\,r\hat{\theta} + [A_\phi/(r\sin\theta)]\,r\sin\theta\,\hat{\phi} \Rightarrow \boxed{(\vec{A}\cdot\vec{\nabla})\vec{r} = \vec{A}}$

(4) (a) $\vec{\omega} = \omega\hat{z}$ and $\vec{r} = \rho\hat{\rho} + z\hat{z} \Rightarrow \boxed{\vec{v} = \vec{\omega}\times\vec{r} = \omega\rho\hat{\phi}}$

$$\vec{\nabla}\times\vec{a} = \left(\frac{1}{\rho}\frac{\partial a_z}{\partial\phi} - \frac{\partial a_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}\right)\hat{\phi} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho a_\phi) - \frac{\partial a_\rho}{\partial\phi}\right]\hat{z}$$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{v} = (2\omega\rho/\rho)\,\hat{z} = 2\omega\hat{z}}$

(b) In spherical coords, $\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$, $\vec{r} = r\hat{r}$

$\Rightarrow \vec{v} = \vec{\omega}\times\vec{r} = -\omega r\sin\theta(\hat{\theta}\times\hat{r}) \Rightarrow \boxed{\vec{v} = \omega r\sin\theta\,\hat{\phi}}$

(c) $$\vec{\nabla}\times\vec{a} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(a_\phi\sin\theta) - \frac{\partial a_\theta}{\partial\phi}\right]\hat{r} + \left[\frac{1}{r\sin\theta}\frac{\partial a_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(r a_\phi)\right]\hat{\theta}$$

$$+ 1/r\left[\partial/\partial r(r a_\theta) - \partial a_r/\partial\theta\right]\hat{\phi}$$

$\Rightarrow \vec{\nabla}\times\vec{v} = (2\omega r\sin\theta\cos\theta/r\sin\theta)\,\hat{r} - (2\omega r\sin\theta/r)\,\hat{\theta}$

$\rightarrow \boxed{\vec{\nabla}\times\vec{v} = 2\omega(\hat{r}\cos\theta - \hat{\theta}\sin\theta) = 2\omega\hat{z}}$

[Text, 2.11] (5) (a) $\psi = zx^2/(x^2+y^2+z^2) = zx^2/r^2$

$\partial\psi/\partial x = 2zx/r^2 - (2zx^2/r^3)(\partial r/\partial x) = 2zx/r^2 - 2zx^3/r^4$

$$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{2z}{r^2} - \frac{4zx^2}{r^4} - \frac{6zx^2}{r^4} + \frac{8zx^4}{r^6} = \frac{2z}{r^2} - 10\frac{x^2z}{r^4} + 8\frac{x^4z}{r^6}$$

$\partial\psi/\partial y = -2zx^2/r^3(\partial r/\partial y) = -2x^2yz/r^4$

$$\Rightarrow \frac{\partial^2\psi}{\partial y^2} = -2\frac{x^2z}{r^4} + 8\frac{x^2y^2z}{r^6}$$

Set 2

5 a (continued)

$$\partial\psi/\partial z = x^2/r^2 - 2zx^2/r^2(\partial r/\partial z) = x^2/r^2 - 2x^2z^2/r^4$$

$$\Rightarrow \frac{\partial^2\psi}{\partial z^2} = -2\frac{x^2z}{r^4} - 4\frac{x^2z}{r^4} + 8\frac{x^2z^3}{r^6} = -6\frac{x^2z}{r^4} + 8\frac{x^2z^3}{r^6}$$

$$\Rightarrow \nabla^2\psi = 2\frac{z}{r^2} - 18\frac{x^2z}{r^4} + 8\frac{(x^2+y^2+z^2)x^2z}{r^6} = 2\frac{z}{r^2} - 10\frac{x^2z}{r^4}$$

b In spherical coords, $\psi = r\sin^2\theta\cos\theta\cos^2\phi$

$$\Rightarrow \partial\psi/\partial r = \sin^2\theta\cos\theta\cos^2\phi \Rightarrow \partial/\partial r(r^2\partial\psi/\partial r) = 2r\sin^2\theta\cos\theta\cos^2\phi$$

$$\partial\psi/\partial\theta = r\cos^2\phi\,(2\sin\theta\cos^2\theta - \sin^3\theta)$$

$$\Rightarrow \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = r\cos^2\phi\,(4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta - 4\sin^3\theta\cos\theta)$$

$$= 4r\sin\theta\cos\theta\cos^2\phi\,(\cos^2\theta - 2\sin^2\theta)$$

$$\partial\psi/\partial\phi = -2r\sin^2\theta\cos\theta\cos\phi\sin\phi$$

$$\Rightarrow \frac{\partial^2\psi}{\partial\phi^2} = 2r\sin^2\theta\cos\theta\,(\sin^2\phi - \cos^2\phi)$$

$$\Rightarrow \nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$= \frac{1}{r}\left[2\sin^2\theta\cos\theta\cos^2\phi + (4\cos^3\theta - 8\cos\theta\sin^2\theta)\cos^2\phi + 2\cos\theta(\sin^2\phi - \cos^2\phi)\right]$$

$$= \frac{2}{r}\cos\theta\left[\sin^2\phi + (2\cos^2\theta - 3\sin^2\theta - 1)\cos^2\phi\right]$$

Now $2\cos^2\theta - 3\sin^2\theta - 1 = 2(\cos^2\theta + \sin^2\theta) - 5\sin^2\theta - 1$

$$= 1 - 5\sin^2\theta$$

$$\Rightarrow \nabla^2\psi = \frac{2}{r}\cos\theta\,(\sin^2\phi + \cos^2\phi) - \frac{10}{r}\cos\theta\sin^2\theta\cos^2\phi$$

$$\Rightarrow \boxed{\nabla^2\psi = \frac{2}{r}\cos\theta - \frac{10}{r}\cos\theta\sin^2\theta\cos^2\phi = 2\frac{z}{r^2} - 10\frac{zx^2}{r^4}}$$

⑥ $\vec{F} = P\left[(2\cos\theta/r^3)\,\hat{r} + (\sin\theta/r^3)\,\hat{\theta}\right]$

ⓐ In spherical coords

$$\vec{\nabla}\times\vec{F} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(F_\phi\sin\theta) - \frac{\partial F_\theta}{\partial\phi}\right]\hat{r} + \left[\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi)\right]\hat{\theta}$$

$$+ {}^1\!/_r\left[{}^\partial\!/_{\partial r}(rF_\theta) - \partial F_r/\partial\theta\right]\hat{\phi}$$

$$= \frac{1}{r}\left(-\frac{2}{r^3}\sin\theta + \frac{2}{r^3}\sin\theta\right)\hat{\phi} = 0 \Rightarrow \boxed{\vec{\nabla}\times\vec{F} = 0}$$

ⓑ $\partial\phi/\partial r = -F_r = -2P\cos\theta/r^3 \Rightarrow \phi = +P\cos\theta/r^2 + f(\theta)$

${}^1\!/_r\,\partial\phi/\partial\theta = -F_\theta = -P\sin\theta/r^3$

$\Rightarrow -P\sin\theta/r^3 + {}^1\!/_r\,df/d\theta = -P\sin\theta/r^3 \Rightarrow f = c$

$\Rightarrow \boxed{\phi = P\cos\theta/r^2 + c}$

[Text 3.23] ⑦ ⓐ $\vec{F} = \alpha r/(r^2+a^2)^{3/2}\ \hat{r}$

On the spherical surface, $\hat{n} = \hat{r}$ and $dS = (\sqrt{3}a)^2\,d(\cos\theta)\,d\phi$

$$\Rightarrow \oint \vec{F}\cdot\hat{n}\,dS = \frac{\sqrt{3}\,\alpha a}{(3a^2+a^2)^{3/2}}(\sqrt{3}a)^2\int_{-1}^{1}d(\cos\theta)\int_0^{2\pi}d\phi = \frac{3\sqrt{3}\,a^3\alpha}{8a^3}4\pi$$

$$\Rightarrow \boxed{\oint \vec{F}\cdot\hat{n}\,dS = {}^3\!/_2\,\pi\sqrt{3}\,\alpha}$$

ⓑ In spherical coords,

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$= {}^1\!/_{r^2}\left[3\alpha r^2/(r^2+a^2)^{3/2} - {}^3\!/_2\left[\alpha r^3/(r^2+a^2)^{5/2}\right](2r)\right]$$

$$= 3\alpha\frac{r^2+a^2-r^2}{(r^2+a^2)^{5/2}} \Rightarrow \boxed{\vec{\nabla}\cdot\vec{F} = \frac{3\alpha a^2}{(r^2+a^2)^{5/2}}}$$

Volume element $d\tau = r^2dr\,d(\cos\theta)\,d\phi$

(7)(b) (continued)

$$\Rightarrow \int \vec{\nabla}\cdot\vec{F}\, d\tau = 3\alpha a^2 \int_0^{\sqrt{3}a} \frac{r^2\, dr}{(r^2+a^2)^{5/2}} \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi$$

In radial integral, define $r = a\tan u \Rightarrow dr = a\, du/\cos^2 u$

and $r^2 + a^2 = a^2(1+\tan^2 u) = a^2/\cos^2 u$

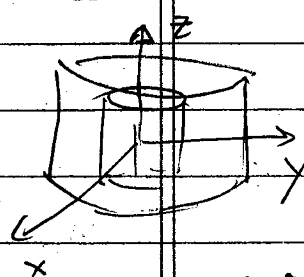
$$\Rightarrow r^2 dr/(r^2+a^2)^{5/2} = (a^3 \tan^2 u\, du/\cos^2 u)/(a^5/\cos^5 u)$$

$$= \sin^2 u \cos u\, du/a^2 = \sin^2 u\, d(\sin u)/a^2$$

$$\Rightarrow \int \vec{\nabla}\cdot\vec{F}\, d\tau = 4\pi \frac{3\alpha a^2}{a^2} \int_0^{\tan^{-1}\sqrt{3}} \sin^2 u\, d(\sin u) = 4\pi\alpha \sin^3 u \Big|_0^{\tan^{-1}\sqrt{3}}$$

Now $\tan^{-1}\sqrt{3} = \pi/3 \Rightarrow \sin(\tan^{-1}\sqrt{3}) = \sqrt{3}/2 \rightarrow \boxed{\int \vec{\nabla}\cdot\vec{F}\, d\tau = \tfrac{3}{2}\pi\sqrt{3}\alpha}$

[Text 3.26] (8)(a) $\vec{F} = F_0(\rho/a)\cos(\lambda z)\,\hat{\rho} + F_0 \sin(\lambda z)\,\hat{z}$



Curved surface at $\rho = a \rightarrow \hat{n} = -\hat{\rho} \Rightarrow \vec{F}\cdot\hat{n} = -F_0\cos(\lambda z)$

Curved surface at $\rho = 2a \rightarrow \hat{n} = +\hat{\rho} \Rightarrow \vec{F}\cdot\hat{n} = 2F_0\cos(\lambda z)$

Flat surfaces at $z = \pm\pi a/2 \rightarrow \hat{n} = \pm\hat{z}$

$$\Rightarrow \vec{F}\cdot\hat{z} = \pm F_0 \sin(\pm\lambda\pi a/2) = F_0\sin(\lambda\pi a/2)$$

$$\Rightarrow \oint \vec{F}\cdot\hat{n}\, dS = F_0 \int_0^{2\pi} d\phi \int_{-\pi a/2}^{\pi a/2} dz\,[-a\cos(\lambda z) + 4a\cos(\lambda z)]$$

$$+ 2F_0 \sin\left(\frac{\pi\lambda a}{2}\right) \int_0^{2\pi} d\phi \int_a^{2a} \rho\, d\rho$$

$$= 6\pi F_0 a \frac{\sin(\lambda z)}{\lambda}\Bigg|_{-\pi a/2}^{\pi a/2} + 4\pi F_0 \sin\left(\frac{\pi\lambda a}{2}\right) \frac{1}{2}\rho^2\Bigg|_a^{2a}$$

$$\rightarrow \boxed{\oint \vec{F}\cdot\hat{n}\, dS = 6\pi F_0 a \sin\left(\frac{\pi\lambda a}{2}\right)\left(\frac{2}{\lambda} + a\right)}$$

(8)(b) $\vec{\nabla}\cdot\vec{F} = 1/\rho\ \partial/\partial\rho(\rho F_\rho) + 1/\rho\ \partial F_\phi/\partial\phi + \partial F_z/\partial z$

$= 2F_0/a\ \cos(\lambda z) + \lambda F_0 \cos(\lambda z) = F_0 \cos(\lambda z)(2/a + \lambda)$

$\Rightarrow \int \vec{\nabla}\cdot\vec{F}\, d\tau = F_0\left(\frac{2}{a}+\lambda\right)\int_0^{2\pi} d\phi \int_a^{2a} \rho\, d\rho \int_{-\pi a/2}^{\pi a/2} \cos(\lambda z)\, dz$

$= F_0\,(2/a+\lambda)(2\pi)(1/2\ 3a^2)\ 1/\lambda\ \sin(\lambda z)\Big|_{-\pi a/2}^{\pi a/2}$

$\rightarrow \boxed{\int \vec{\nabla}\cdot\vec{F}\, d\tau = 6\pi F_0 a \left(\frac{2}{\lambda}+a\right)\sin\left(\frac{\pi\lambda a}{2}\right)}$

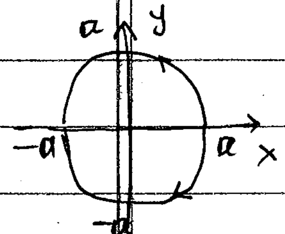
(9)(a) $\vec{A} = (-y\,\hat{x} + x\,\hat{y})/\rho$

$\vec{\nabla}\times\vec{A} = (\partial A_y/\partial x - \partial A_x/\partial y)\,\hat{z}$

$= (1/\rho - x^2/\rho^3 + 1/\rho - y^2/\rho^3)\,\hat{z} = \hat{z}/\rho \Rightarrow \boxed{\vec{\nabla}\times\vec{A} = \hat{z}/\rho}$

(b) $y = \rho\sin\phi$ and $x = \rho\cos\phi \Rightarrow \vec{A} = -\hat{x}\sin\phi + \hat{y}\cos\phi = \hat{\phi}$

$\Rightarrow \boxed{\vec{A} = \hat{\phi}}$



For semicircles of radius a, $\vec{dr} = a\,d\phi\,\hat{\phi}$

$\Rightarrow \vec{A}\cdot\vec{dr} = a\,d\phi$

upper ½-plane $\rightarrow \int \vec{A}\cdot\vec{dr} = \int_0^{\pi} a\,d\phi = \underline{\underline{\pi a}}$

lower ½-plane $\rightarrow \int \vec{A}\cdot\vec{dr} = \int_0^{-\pi} a\,d\phi = \underline{\underline{-\pi a}}$

(c) In Cartesian coords, $\vec{dr} = \hat{x}\,dx + \hat{y}\,dy$

$\Rightarrow \vec{A}\cdot\vec{dr} = (-y\,dx + x\,dy)/a$

Also $x^2 + y^2 = a^2 \Rightarrow y\,dy = -x\,dx \Rightarrow dy = -(x/y)\,dx$

$\Rightarrow \int \vec{A}\cdot\vec{dr} = -\int\left(y + \frac{x^2}{y}\right)dx = -\int \frac{x^2+y^2}{y}\,dx = -\int \frac{a^2\,dx}{\sqrt{a^2-x^2}}$

in the upper half plane where $y > 0$ $\rightarrow$ choose $a = 1$

$\Rightarrow \int \vec{A}\cdot\vec{dr} = -\int_{+1}^{-1} \frac{dx}{\sqrt{1-x^2}} = \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} \Rightarrow \boxed{\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \pi}$