Set 3 — Solutions —

Text 1.9 **1** (a) $C = -\tfrac{i}{2}(AB-BA) \Rightarrow C^2 = -\tfrac{1}{4}(AB-BA)(AB-BA)$

$= \tfrac{1}{4}\left(AB^2A + BA^2B - ABAB - BABA\right)$

Now $BA^2B = B^2 = I$ and $AB^2A = A^2 = I$

$ABAB = -A^2B^2 = -I$ and $BABA = -B^2A^2 = -I \Rightarrow \underline{\underline{C^2 = I}}$

$[B,C] = BC - CB = \tfrac{i}{2}\left[B(BA-AB) + (AB-BA)B\right]$

$= \tfrac{i}{2}\left(B^2A - BAB + AB^2 - BAB\right)$

$= \tfrac{i}{2}\left(A + B^2A + A + AB^2\right) = \tfrac{i}{2}(4A) \Rightarrow \underline{\underline{[B,C] = 2iA}}$

(b) $AC = \tfrac{i}{2}A(BA-AB) = \tfrac{i}{2}(-BAA + ABA) = \tfrac{i}{2}(AB-BA)A = \underline{-CA}$

Now $[[A,B],[B,C]] = [2iC, 2iA] = -4(CA - AC)$

$\Rightarrow \underline{\underline{[[[A,B],[B,C]],[A,B]]}} = -4[CA-AC, 2iC] = -8i(CAC - ACC - CCA + CAC)$

$= -8i(-AC^2 - AC^2 - C^2A - AC^2) = -8i(-4A) = \underline{\underline{32iA}}$

2 (a) ij element of $C \rightarrow C_{ij} = \sum_k (A_{ik}B_{kj} - B_{ik}A_{kj})$

$\Rightarrow Tr(C) = \sum_i C_{ii} = \sum_{ik}(A_{ik}B_{ki} - B_{ik}A_{ki})$

$= \sum_{ik} A_{ik}B_{ki} - \sum_{ik} A_{ki}B_{ik} = 0 \Rightarrow \underline{\underline{Tr(C) = 0}}$

(b) $BA = AB \Rightarrow \sum_k B_{ik}A_{kj} = \sum_k A_{ik}B_{kj}$

But $A_{kj} = A_{jj}\delta_{kj}$ and $A_{ik} = A_{ii}\delta_{ik}$ since A diagonal

$\Rightarrow B_{ij}A_{jj} = A_{ii}B_{ij}$

But $A_{jj} \neq A_{ii}$ unless $j = i \Rightarrow B_{ij} = 0$ unless $j = i$

$\rightarrow$ $\underline{\underline{B \text{ diagonal}}}$

3 $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix} \rightarrow \det A = 3(8-1) - 2(8-1) + 1(2-2) = 7$

Cofactors $C_{11} = 7$, $C_{22} = 11$, $C_{33} = 2$

$C_{12} = C_{21} = -7$, $C_{13} = C_{31} = 0$, $C_{23} = C_{32} = -1$

(3) (continued)

$$\Rightarrow \boxed{A^{-1} = \frac{1}{7}\begin{pmatrix} 7 & -7 & 0 \\ -7 & 11 & -1 \\ 0 & -1 & 2 \end{pmatrix}} \rightarrow \text{check: } AA^{-1} = I$$

(4) (a) $C = S^{+}S \Rightarrow Tr(C) = \sum_i C_{ii} = \sum_{ik} (S^{+})_{ik} S_{ki}$

$$= \sum_{ik} (S^{*})_{ki} S_{ki} = \sum_{ik} |S_{ki}|^2$$

But $|S_{ki}|^2 \geq 0$ and at least one $S_{ki} \neq 0 \Rightarrow \sum_{ik} |S_{ki}|^2 > 0$

$\Rightarrow \underline{\underline{Tr(C) > 0}}$

(b) A, B Hermitian $\Rightarrow A^{+} = A$ and $B^{+} = B$

$C = AB + BA \Rightarrow C^{+} = B^{+}A^{+} + A^{+}B^{+} = BA + AB = C$

$\Rightarrow \underline{\underline{C \text{ Hermitian}}}$

$D = i(AB - BA) \Rightarrow D^{+} = -i(B^{+}A^{+} - A^{+}B^{+}) = -i(BA - AB) = D$

$\Rightarrow \underline{\underline{D \text{ Hermitian}}}$

[Text 1.14] (5) (a) Coeff. matrix $= \begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 3 \end{pmatrix} \rightarrow \det = 3(-11) - 2(-1) + 1(7) \neq 0$

$\rightarrow$ <u>no solution except the trivial one</u>

(b) Coeff. matrix $= \begin{pmatrix} 2 & -b & -b \\ 1 & -2a & 2a \\ 1 & b-6a & b+6a \end{pmatrix}$

$\rightarrow \det = 2(2a)(-2b) + b(b+4a) - b(b-4a) = -8ab + b^2 + 4ab - b^2 + 4ab = 0$

$\rightarrow$ non-trivial soln exists

$$\frac{y}{x} = \frac{a_{13}a_{21} - a_{11}a_{23}}{a_{12}a_{23} - a_{13}a_{22}} = \frac{-(b+4a)}{-4ab}, \quad \frac{z}{x} = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{12}a_{23} - a_{13}a_{22}} = \frac{b-4a}{-4ab}$$

$$\Rightarrow \boxed{y = \frac{b+4a}{4ab}\, x \qquad z = \frac{-b+4a}{4ab}\, x, \qquad x \text{ arbitrary}}$$

[Text 1.16] ⑥ Coeff. matrix $\to \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 3 & 4 \end{pmatrix} \to \det = 1(1) - 2(7) + 3(5) = 2$

$\Rightarrow$ unique soln exists since $\det \neq 0$

$\to x_1 = \frac{1}{2} \det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 3 & 4 \end{pmatrix} = \frac{1}{2}\left[1(1) - 2(-7) + 3(-6)\right] = -\frac{3}{2} \to \boxed{x_1 = -\frac{3}{2}}$

$x_2 = \frac{1}{2} \det \begin{pmatrix} 1 & 1 & 3 \\ 3 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix} = \frac{1}{2}\left[1(-7) - 1(7) + 3(7)\right] = \frac{7}{2} \to \boxed{x_2 = \frac{7}{2}}$

$x_3 = \frac{1}{2} \det \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 1 & 3 & 3 \end{pmatrix} = \frac{1}{2}\left[1(6) - 2(7) + 1(5)\right] = -\frac{3}{2} \Rightarrow \boxed{x_3 = -\frac{3}{2}}$

[Text 1.7] ⑦ ⓐ A Hermitian $\Rightarrow A^+ = A$, U unitary $\Rightarrow U^+ = U^{-1}$

$(U^{-1}AU)^+ = U^+A^+(U^{-1})^+ = U^{-1}AU \Rightarrow$ $\underline{U^{-1}AU \text{ Hermitian}}$

ⓑ A anti-Hermitian $\Rightarrow A^+ = -A \Rightarrow (iA)^+ = -iA^+ = iA$

$\Rightarrow$ $\underline{iA \text{ Hermitian}}$

ⓒ AB Hermitian $\Rightarrow (AB)^+ = AB \Rightarrow B^+A^+ = AB \Rightarrow BA = AB$

since A, B Hermitian $\to$ run proof in reverse to prove

$[A, B] = 0 \Rightarrow AB$ Hermitian

ⓓ $A = (1-S)(1+S)^{-1} \Rightarrow (1+S)A = (1+S)(1-S)(1+S)^{-1} = (1-S)(1+S)(1+S)^{-1}$

$= 1 - S \Rightarrow \left[(1+S)A\right]^T = 1 - S^T \Rightarrow A^T(1+S^T) = 1 - S^T$

$\Rightarrow A^T(1-S) = 1+S$ (since $S^T = -S$) $\Rightarrow$ $\underline{A^T = (1+S)(1-S)^{-1}}$

$\Rightarrow A^TA = (1+S)(1-S)^{-1}(1-S)(1+S)^{-1} = (1+S)(1+S)^{-1} = I \Rightarrow A^T = A^{-1}$

$\Rightarrow$ $\underline{A \text{ orthogonal}}$

S antisymmetric $\Rightarrow S = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \Rightarrow 1 \pm S = \begin{pmatrix} 1 & \pm a \\ \mp a & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & -a \\ a & 1 \end{pmatrix} \Rightarrow \begin{cases} \cos\theta - a\sin\theta = 1 \\ \sin\theta + a\cos\theta = -a \end{cases}$

⑦ ⓓ (continued)

$\Rightarrow a\sin^2\theta + a\cos^2\theta = -\sin\theta - a\cos\theta \Rightarrow \boxed{a = -\sin\theta/(1+\cos\theta)}$

ⓔ same proof as in part d with transpose replaced by Hermitian conjugate

[Text 1.18] ⑧ ~~Coeff matrix~~ $= \begin{pmatrix} 1 & \alpha & 0 \\ 1 & -1 & 3 \\ 2 & -2 & \alpha \end{pmatrix} \rightarrow$ ~~$\det = 6-\alpha-\alpha(\alpha-6) = (\alpha+1)(6-\alpha)$~~

$\Rightarrow$ unique soln requires $\alpha \neq -1$ and $\alpha \neq 6$

ⓐ For unique soln ($\alpha \neq -1$, $\alpha \neq 6$)

$x_1 + \alpha x_2 = 1$ (1)

$x_1 - x_2 + 3x_3 = -1$ (2)

$2x_1 - 2x_2 + \alpha x_3 = -2$ (3)

$2 \times$ eq(2) $-$ eq(3) $\Rightarrow (6-\alpha)x_3 = 0 \Rightarrow \boxed{x_3 = 0}$

Eq (1) $\Rightarrow x_1 = 1 - \alpha x_2 \rightarrow$ insert into eq. (2)

$\rightarrow 1 - \alpha x_2 - x_2 = -1 \Rightarrow \boxed{x_2 = 2/(1+\alpha)}$

$\Rightarrow$ ~~$x_1 = 1 - \alpha x_2$~~ $= \dfrac{1+\alpha-2\alpha}{1+\alpha} \Rightarrow \boxed{x_1 = \dfrac{1-\alpha}{1+\alpha}}$

ⓑ No soln $\rightarrow$ $\alpha = -1$

Eq (1) $\Rightarrow x_1 - x_2 = 1$

Eq (2) $\Rightarrow x_1 - x_2 = -1$ (since $x_3 = 0$ for $\alpha \neq 6$)

$\rightarrow$ contradictory eqns $\Rightarrow$ no solution

ⓒ infinite solns $\rightarrow$ $\alpha = 6$ $\Rightarrow$ x_3 arbitrary

Eq (1) $\Rightarrow x_1 + 6x_2 = 1 \Rightarrow x_1 = 1 - 6x_2$

Eq (2) $\Rightarrow 1 - 6x_2 - x_2 + 3x_3 = -1 \Rightarrow x_2 = \frac{1}{7}(3x_3 + 2)$

$\Rightarrow x_1 = 1 - 6x_2 = -\frac{1}{7}(18x_3 + 5)$