Set 4 — Solutions —

① ⓐ Let A', B' be the diagonalized matrices corresp. to A, B

$\Rightarrow A' = RAR^{-1}$, $B' = RBR^{-1}$ and $B'A' = A'B'$

(since diag. matrices always commute)

$\Rightarrow (RBR^{-1})(RAR^{-1}) = (RAR^{-1})(RBR^{-1}) \Rightarrow RBAR^{-1} = RABR^{-1}$

$\Rightarrow BA = AB \Rightarrow$ $\underline{A, B \text{ commute}}$

ⓑ Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Rightarrow (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = 0$

$\Rightarrow \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$

$Tr(A) = a_{11} + a_{12}$ and $\det(A) = a_{11}a_{22} - a_{12}a_{21} \Rightarrow \underline{\underline{\lambda^2 - \lambda Tr(A) + \det(A) = 0}}$

② ⓐ $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow (\lambda-1)[\lambda(\lambda-1)-1] - (\lambda-1) = (\lambda-1)(\lambda^2 - \lambda - 2)$

$= (\lambda-1)(\lambda-2)(\lambda+1) = 0 \Rightarrow \underline{\underline{\lambda = 1, 2, -1}}$

Now $x + y = \lambda x$ (1), $x + z = \lambda y$ (2), $y + z = \lambda z$ (3)

Eq(1) − Eq(3) $\rightarrow (x - z) = \lambda(x - z) \Rightarrow \underline{z = x \quad \text{if } \lambda \neq 1} \Rightarrow \underline{y = 2x/\lambda}$

$\underline{\text{For } \lambda = 1}$, Eq(1) $\Rightarrow \underline{y = 0} \longrightarrow$ Eq(2) $\Rightarrow \underline{z = -x}$

$\Rightarrow$ eigenvectors $\underline{\underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} (\lambda = 1)}}$ $\underline{\underline{\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} (\lambda = -1)}}$ $\underline{\underline{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (\lambda = 2)}}$

ⓑ $A = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix} \Rightarrow (\lambda-5)(\lambda-3)^2 - 3(\lambda-3) = (\lambda-3)(\lambda^2 - 8\lambda + 12)$

$= (\lambda-3)(\lambda-6)(\lambda-2) = 0 \Rightarrow \underline{\underline{\lambda = 3, 6, 2}}$

Now $5x + \sqrt{3}z = \lambda x$ (1) $3y = \lambda y$ (2) $\sqrt{3}x + 3z = \lambda z$ (3)

Eq(2) $\Rightarrow \underline{y = 0 \text{ if } \lambda \neq 3}$, Eq(3) $\Rightarrow \underline{z = \sqrt{3}/(\lambda-3)\ x}$

$\underline{\text{For } \lambda = 3}$, $\underline{\text{Eq(3)} \Rightarrow x = 0}$ and $\underline{\text{Eq(1)} \Rightarrow z = 0}$

→ eigenvectors $\underline{\underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} (\lambda = 3)}}$ $\underline{\underline{\frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix} (\lambda = 6)}}$ $\underline{\underline{\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -\sqrt{3} \end{pmatrix} (\lambda = 2)}}$

2 (c)

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \to (\lambda-1)\left[(\lambda-1)^2 - 1\right] - (\lambda-1) + 1 + 1 - (\lambda-1)$$

$$= (\lambda-1)^3 - 3(\lambda-1) + 2 = \lambda^3 - 3\lambda^2 + 4 = (\lambda+1)(\lambda^2 - 4\lambda + 4)$$

$$= (\lambda+1)(\lambda-2)^2 \Rightarrow \underline{\lambda = -1 \text{ or } \lambda = 2 \text{ (degenerate)}}$$

Now $x - y - z = \lambda x$ (1) $\quad -x + y - z = \lambda y$ (2) $\quad -x - y + z = \lambda z$ (3)

Eq(1)+(2) $\Rightarrow -2z = \lambda(x+y) \Rightarrow \underline{z = -\lambda/2\,(x+y)}$

$\to$ substitute into Eq (1) $\to (1+\lambda/2)x - (1-\lambda/2)y = \lambda x \Rightarrow \underline{y = x \text{ if } \lambda \neq 2}$

For $\lambda = 2$, x and y are both arbitrary

$\to$ eigenvectors $\underline{\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (\lambda = -1)}$ $\quad \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ or } \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} (\lambda = 2)}$

[Text 1.23] 3

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \to \lambda^2(\lambda-1) - \lambda - 1 = (\lambda-1)(\lambda^2-1) = (\lambda-1)^2(\lambda+1) = 0$$

$$\Rightarrow \underline{\lambda = -1 \text{ or } \lambda = 1 \text{ (degenerate)}}$$

Now $z = \lambda x$ (1) $\quad y = \lambda y$ (2) $\quad x = \lambda z$ (3)

Eq(2) $\Rightarrow \underline{y = 0 \text{ if } \lambda \neq 1}$ $\quad$ Eq(1) + Eq(3) $\Rightarrow \underline{x + z = 0 \text{ if } \lambda \neq 1}$

For $\lambda = 1$, $\underline{y \text{ arbitrary}}$ and $\underline{z = x}$

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \lambda(\lambda^2-1) - \lambda - 1 - 1 - \lambda = \lambda^3 - 3\lambda - 2 = (\lambda+1)(\lambda^2 - \lambda - 2)$$

$$= (\lambda+1)(\lambda+1)(\lambda-2) = 0 \Rightarrow \underline{\lambda = 2 \text{ or } \lambda = -1 \text{ (degenerate)}}$$

Now $y + z = \lambda x$ (1) $\quad x + z = \lambda y$ (2) $\quad x + y = \lambda z$ (3)

Eq(1) − Eq(2) $\Rightarrow y - x = \lambda(x-y) \Rightarrow \underline{y = x \text{ if } \lambda \neq -1} \Rightarrow \underline{z = x \text{ for } \lambda = 2}$

$\underline{\text{For } \lambda = -1}$, $\underline{z = -(x+y)}$ and $\underline{y \text{ arbitrary}}$

$\to$ choose $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\underline{\lambda_A = -1 \quad \lambda_B = -1}$ $\quad \underline{\lambda_A = 1 \quad \lambda_B = 2}$ $\quad \underline{\lambda_A = 1 \quad \lambda_B = -1}$

[Text 1.27] 4

$H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix} \rightarrow (\lambda-10)(\lambda-2) - 9 = \lambda^2 - 12\lambda + 11 = (\lambda-11)(\lambda-1) = 0$

$\Rightarrow \underline{\lambda = 1 \text{ or } \lambda = 11}$

Now $10x + 3iy = \lambda x \Rightarrow y = -i/3\,(\lambda - 10)x$

$\rightarrow$ eigenvectors $\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3i \end{pmatrix}$ $(\lambda = 1)$ $\frac{1}{\sqrt{10}}\begin{pmatrix} 3i \\ 1 \end{pmatrix}$ $\lambda = 11$

$\Rightarrow \boxed{U = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}} \Rightarrow U^+ = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix}$

$\rightarrow U^+ H U = \frac{1}{10}\begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix}\begin{pmatrix} 1 & 33i \\ 3i & 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix}$ as required

5 a $A^T A \vec{f} = A^T \lambda \vec{g} = \lambda A^T \vec{g} = \lambda^2 \vec{f} \Rightarrow \underline{(A^T A)\vec{f} = \lambda^2 \vec{f}}$

b $A A^T \vec{g} = A \lambda \vec{f} = \lambda A \vec{f} = \lambda^2 \vec{g} \Rightarrow \underline{(A A^T)\vec{g} = \lambda^2 \vec{g}}$

c $A = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 2 \\ 1 & -4 \end{pmatrix} \Rightarrow A^T = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix}$

$\Rightarrow A^T A = \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 20 \end{pmatrix} \rightarrow (\lambda^2 - 1)(\lambda^2 - 4) = 0 \Rightarrow \underline{\lambda^2 = 1 \text{ or } \lambda^2 = 4}$

$\Rightarrow \underline{\vec{f} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ (\lambda^2 = 1)}$ $\underline{\vec{f} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ (\lambda^2 = 4)}$

$A A^T = \frac{1}{5}\begin{pmatrix} 8 & -6 \\ -6 & 17 \end{pmatrix} \rightarrow (\lambda - 8/5)(\lambda - 17/5) - \frac{36}{25} = \lambda^2 - 5\lambda + 4 = 0$

$\Rightarrow \lambda^2 = 1 \text{ or } \lambda^2 = 4$

Now $8/5\, x - 6/5\, y = \lambda x \Rightarrow y = [(8 - 5\lambda)/6]\, x$

$\Rightarrow \underline{\vec{g} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \ (\lambda^2 = 1)}$ $\underline{\vec{g} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \ (\lambda^2 = 4)}$

[Text 1.31] (6) For a 2x2 symmetric matrix A with $a_{21} = a_{12}$

$$\vec{x}\cdot A\cdot\vec{x} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2$$

$$\Rightarrow A = \begin{pmatrix} 8 & -3 \\ -3 & 8 \end{pmatrix} \to (\lambda-8)^2 - 9 = \lambda^2 - 16\lambda + 55 = (\lambda-5)(\lambda-11) = 0$$

$\Rightarrow \underline{\lambda = 5 \text{ or } 11} \to$ conic section is an ellipse (both $\lambda > 0$)

Now $8x - 3y = \lambda x \Rightarrow \underline{y = (8-\lambda)/3\ x}$

eigenvectors $\to \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}(\lambda=5)$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}(\lambda=11)$

$\lambda = 5 \to$	major axis	$\to$	45° in the 1st quadrant / 4th quadrant
$\lambda = 11 \to$	minor axis	$\to$	45° in the 2nd quadrant / 3rd quadrant

In the rotated coord. system, $5x'^2 + 11y'^2 = 110$

$\Rightarrow x'^2/(110/5) + y'^2/(110/11) = 1 \Rightarrow \boxed{a = \sqrt{110/5} = \sqrt{22} \text{ (major)}}$

and $\boxed{b = \sqrt{110/11} = \sqrt{10} \text{ (minor)}}$

[Text 1.32] (7) For a 3x3 symmetric matrix A with $a_{21} = a_{12}$, $a_{31} = a_{13}$, $a_{32} = a_{23}$

$$\vec{x}\cdot\vec{A}\cdot\vec{x} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2$$

$$\Rightarrow A = \begin{pmatrix} 5 & -5 & 1 \\ -5 & 11 & -5 \\ 1 & -5 & 5 \end{pmatrix} \to (\lambda-5)[(\lambda-11)(\lambda-5) - 25] - 25(\lambda-5) - 25 - 25 - (\lambda-11)$$

$$= \lambda^3 - 21\lambda^2 + 84\lambda - 64 = (\lambda-1)(\lambda^2 - 20\lambda + 64) = (\lambda-1)(\lambda-4)(\lambda-16) = 0$$

$\Rightarrow \underline{\lambda = 1, 4, \text{ or } 16}$

Now $5x - 5y + z = \lambda x$ (1) $\quad -5x + 11y - 5z = \lambda y$ (2) $\quad x - 5y + 5z = \lambda z$ (3)

Eq (1) − Eq (3) $\to 4(x-z) = \lambda(x-z) \Rightarrow z = x$ if $\lambda \neq 4 \Rightarrow y = 10x/(11-\lambda)$

So for $\lambda = 1$, $x = y = z \to$ direction of the longest axis

In the rotated system $x'^2 + 4y'^2 + 16z'^2 = 4$

$\Rightarrow x'^2/4 + y'^2 + z'^2/(1/4) = 1 \to \boxed{\text{axis lengths} = 2, 1, \text{ and } 1/2}$

Set 4

[Text 1.33] (8) For a 3×3 symmetric matrix A with $a_{21}=a_{12}$, $a_{31}=a_{13}$, $a_{32}=a_{23}$

$$\vec{x}\cdot A\cdot\vec{x} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{33}x_3^2$$

$$\Rightarrow A = \begin{pmatrix} 7 & 10 & -10 \\ 10 & 7 & -10 \\ -10 & -10 & 7 \end{pmatrix} \rightarrow (\lambda-7)\left[(\lambda-7)^2-100\right] - 100(\lambda-7) - 1000 - 1000 - 100(\lambda-7)$$

$$= (\lambda-7)^3 - 300(\lambda-7) - 2000 = \lambda^3 - 21\lambda^2 - 153\lambda - 243$$

$$= (\lambda-27)(\lambda^2+6\lambda+9) = (\lambda-27)(\lambda+3)^2 = 0 \Rightarrow \lambda = 27, -3 \text{ (degenerate}$$

→ symmetry axis in the direction of the $\lambda = 27$ eigenvector

Now $7x + 10y - 10z = 27x$ (1) $\quad 10x + 7y - 10z = 27y$ (2)

$-10x - 10y + 7z = 27z$ (3)

Eq(1) − Eq(2) → $-3x + 3y = 27x - 27y \Rightarrow \underline{y = x}$ → Eq(3) ⇒ $\underline{z = -x}$

⇒ **symmetry axis along direction given by $y = x = -z$**