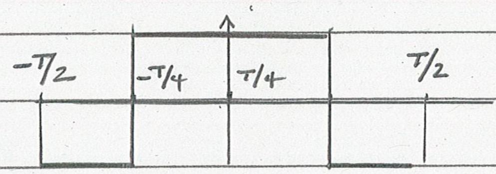
Set 5 — Solutions —

[Text 4.4] 1 (a)



Symmetric fcn $\rightarrow b_n = 0$

$$a_0 = \frac{2}{T}\int f(t)\,dt = 0$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt = \frac{4}{T}\int_0^{T/4}\cos\left(\frac{2\pi n t}{T}\right)dt - \frac{4}{T}\int_{T/4}^{T/2}\cos\left(\frac{2\pi n t}{T}\right)dt$$

$$= \frac{2}{\pi n}\left[\sin\left(\frac{2\pi n t}{T}\right)\Big|_0^{T/4} - \sin\left(\frac{2\pi n t}{T}\right)\Big|_{T/4}^{T/2}\right] = \frac{4}{\pi n}\sin\left(\frac{n\pi}{2}\right)$$

$= 0$ if n even, $= (-)^{(n-1)/2}(4/\pi n)$ if n odd

$$\Rightarrow \boxed{f(t) = \frac{4}{\pi}\sum_{\text{odd } n}\frac{(-)^{(n-1)/2}}{n}\cos\left(\frac{2\pi n t}{T}\right)}$$

(b) Eq (4.60) $\rightarrow F(t') = \frac{4}{\pi}\sum_{\text{odd } n}\frac{1}{n}\sin\left(\frac{2\pi n t'}{T}\right)$

Now. $t' = t + T/4 \rightarrow \sin(2\pi n t'/T) = \sin[2\pi n(t/T + 1/4)]$

$= \sin(n\pi/2)\cos(2\pi n t/T) = (-)^{(n-1)/2}\cos(2\pi n t/T)$ for n odd

$$\Rightarrow \underline{F(t') = f(t)}$$

[Text 4.5] 2 $f(x) = x$ for $-\pi \le x \le \pi \rightarrow$ antisymm $\rightarrow a_n = 0$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$$= \frac{2}{\pi}\left(-\frac{x\cos(nx)}{n}\right)\Big|_0^{\pi} + \cancelto{0}{\frac{2}{n\pi}\int_0^{\pi}\cos(nx)\,dx} = -\frac{2}{n}\cos(n\pi) = (-)^{n+1}\frac{2}{n}$$

$$\Rightarrow \boxed{f(x) = 2\sum_n\left[(-)^{n+1}/n\right]\sin(nx)}$$

For $x = \pi/2$, $f(x) = \pi/2$ and $\sin(nx) = (-)^{(n-1)/2}$ for n odd

$$\Rightarrow \frac{\pi}{2} = 2\sum_{\text{odd } n}\frac{(-)^{n+1+(n-1)/2}}{n} = 2\sum_{n\,\text{odd}}\frac{(-)^{(n-1)/2}}{n} \Rightarrow \boxed{\sum_{n\,\text{odd}}\frac{(-)^{(n-1)/2}}{n} = \frac{\pi}{4}}$$

③ⓐ $f(x) = x^2$ for $-\pi \le x \le \pi \rightarrow$ symmetric $\rightarrow b_n = 0$

$$\rightarrow a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{3}\pi^2\,, \quad a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx$$

$$= \cancelto{0}{\frac{2}{n\pi}x^2\sin(nx)\Big|_0^{\pi}} - \frac{4}{n\pi}\int_0^{\pi} x\sin(nx)dx = \frac{4}{n^2\pi}x\cos(nx)\Big|_0^{\pi} - \cancelto{0}{\frac{4}{n^2\pi}\int_0^{\pi}\cos(nx)dx}$$

$$= (4/n^2)\cos(n\pi) = (-)^n 4/n^2$$

$$\Rightarrow \boxed{f(x) = \frac{\pi^2}{3} + 4\sum \frac{(-)^n}{n^2}\cos(nx)}$$

ⓑ For $x = 0$, $f(x) = 0$ and $\cos(nx) = 1$

$$\Rightarrow 0 = \frac{\pi^2}{3} + 4\sum\frac{(-)^n}{n^2} \Rightarrow \boxed{\sum\frac{(-)^{n+1}}{n^2} = \frac{\pi^2}{12}}$$

④ⓐ $f(x) = 4x(1+x)$, $-1 \le x \le 0$ and $f(x) = 4x(1-x)$, $0 \le x \le 1$

$\rightarrow$ antisymmetric $\rightarrow a_n = 0$

$$\rightarrow b_n = \frac{2}{2}\int_{-1}^{1} f(x)\sin(n\pi x)dx = 8\int_0^1 x(1-x)\sin(n\pi x)dx$$

$$= \cancelto{0}{-\frac{8}{n\pi}x(1-x)\cos(n\pi x)\Big|_0^1} + \frac{8}{n\pi}\int_0^1(1-2x)\cos(n\pi x)dx$$

$$= \cancelto{0}{\frac{8}{(n\pi)^2}(1-2x)\sin(n\pi x)\Big|_0^1} + \frac{16}{(n\pi)^2}\int_0^1 \sin(n\pi x)dx = -\frac{16}{(n\pi)^3}\cos(n\pi x)\Big|_0^1$$

$$= 16/(n\pi)^3[1-\cos(n\pi)] = 32/(n\pi)^3 \text{ for } n \text{ odd}, \quad = 0 \text{ for } n \text{ even}$$

$$\Rightarrow \boxed{f(x) = \frac{32}{\pi^3}\sum_{n\text{ odd}}\frac{\sin(n\pi x)}{n^3}}$$

ⓑ For $x = 1/2$, $f(x) = 1$, $\sin(n\pi x) = (-)^{(n-1)/2}$

$$\Rightarrow \boxed{\sum_{n\text{ odd}}\frac{(-)^{(n-1)/2}}{n^3} = \frac{\pi^3}{32}}$$

[Text 4.20] 5 $f(\theta) = |\sin\theta|$ for $-\pi \le \theta \le \pi \to$ symmetric $\to b_n = 0$

$a_0 = \frac{4}{2\pi}\int_0^\pi \sin\theta \, d\theta = \frac{4}{\pi}$, $a_n = \frac{4}{2\pi}\int_0^\pi \sin\theta \cos(n\theta) \, d\theta$

$= \frac{1}{\pi}\int_0^\pi \{\sin[(n+1)\theta] - \sin[(n-1)\theta]\} \, d\theta = \frac{1}{\pi}\left[\frac{\cos(n-1)\theta}{n-1} - \frac{\cos(n+1)\theta}{n+1}\right]_0^\pi \; (n \neq 1)$

$= \frac{2}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right) = -\frac{4}{\pi}\frac{1}{n^2-1}$ for n even

[Note that $a_1 = {}^2/_\pi \int_0^\pi \sin\theta\cos\theta \, d\theta = {}^1/_\pi \int_0^\pi \sin(2\theta) \, d\theta = 0$]

$$\Rightarrow \boxed{f(\theta) = \frac{2}{\pi} - \frac{4}{\pi}\sum_m \frac{\cos(2m\theta)}{[(2m)^2-1]}} \qquad \left(m = \frac{n}{2}\right)$$

For $\theta = 0$, $f(\theta) = 0$ and $\cos(2m\theta) = 1$

$$\Rightarrow 0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_m \frac{1}{4m^2-1} \Rightarrow \boxed{\sum_m \frac{1}{4m^2-1} = \frac{1}{2}}$$

For $\theta = \pi/2$, $f(\theta) = 1$ and $\cos(2m\theta) = (-)^m$

$$\Rightarrow 1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_m \frac{(-)^m}{4m^2-1} \Rightarrow \sum_m \frac{(-)^m}{4m^2-1} = \frac{1}{2} - \frac{\pi}{4}$$

Now $\underline{\sum_m \frac{1}{16m^2-1}} = \sum_m \frac{1}{4(2m)^2-1} = \frac{1}{2}\sum_n \frac{1+(-1)^n}{4n^2-1} = \underline{\frac{1}{2} - \frac{\pi}{8}}$

[Text 4.14] 6 $f(x) = |x|$ for $-\pi \le x \le \pi \to$ symmetric $\to b_n = 0$

$a_0 = \frac{2}{2\pi}\int_{-\pi}^\pi |x| \, dx = \frac{4}{2\pi}\int_0^\pi x \, dx = \pi$

$a_n = \frac{2}{\pi}\int_0^\pi x\cos(nx) \, dx = \underbrace{\cancel{\frac{2}{n\pi} x \sin(nx)\Big|_0^\pi}}_{0} - \frac{2}{n\pi}\int_0^\pi \sin(nx) \, dx$

$= + {}^2/_{n^2\pi} \cos(nx)\Big|_0^\pi = -{}^2/_{n^2\pi}[1 - \cos(n\pi)] = -{}^4/_{n^2\pi}$ for n odd

⑥ [continued]

$$\rightarrow \boxed{f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_m \frac{\cos[(2m+1)x]}{(2m+1)^2}}$$

Integrating gives

$$\frac{1}{2}x^2 = \frac{\pi}{2}x - \frac{4}{\pi}\sum_m \frac{\sin[(2m+1)x]}{(2m+1)^3} + C$$

To find C, note that at $x=0$, $\sin[(2m+1)x] = 0 \Rightarrow \underline{\underline{C=0}}$

For $x = \pi/2$, $\sin[(2m+1)x] = (-)^m$

$$\Rightarrow \frac{\pi^2}{8} = \frac{\pi^2}{4} - \frac{4}{\pi}\sum_m \frac{(-)^m}{(2m+1)^3} \Rightarrow \boxed{\sum_m \frac{(-)^m}{(2m+1)^3} = \frac{\pi^3}{32}}$$

⑦ ⓐ $a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi}\delta(x)dx = \frac{1}{\pi}$, $a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi}\delta(x)\cos(nx)dx = \frac{1}{\pi}$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi}\delta(x)\sin(nx) = 0 \Rightarrow \boxed{\delta(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_n \cos(nx)}$$

ⓑ $\int \delta(x)dx = \frac{x}{2\pi} + \frac{1}{\pi}\sum_n \frac{\sin(nx)}{n} + C$

Using the result of problem 2 [text 4.5]

$$x = 2\sum_n \frac{(-)^{n+1}}{n}\sin(nx) \Rightarrow \int \delta(x)dx = \frac{1}{\pi}\sum_n \frac{1-(-)^n}{n}\sin(nx) + C$$

$$\Rightarrow \boxed{\int \delta(x)dx = \frac{2}{\pi}\sum_{n\ \mathrm{odd}} \frac{\sin(nx)}{n} + C}$$

ⓒ $f(x) = 0$ for $-\pi \le x \le 0$ and $f(x) = 1$ for $0 \le x \le \pi$

$$\rightarrow a_0 = \frac{2}{2\pi}\int_0^{\pi}dx = 1, \quad a_n = \frac{1}{\pi}\int_0^{\pi}\cos(nx)dx = \frac{1}{\pi n}\sin(nx)\Big|_0^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin(nx)dx = -\frac{1}{\pi n}\cos(nx)\Big|_0^{\pi} = \frac{2}{\pi n} \text{ for } n \text{ odd}$$

⑦ⓒ [continued]

$$\Rightarrow \underline{\underline{f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n\,odd} \frac{\sin(nx)}{n}}} \longrightarrow \boxed{c = \frac{1}{2}}$$

[Text 4.21] ⑧ $f(x) = \cosh(x)$ for $-\pi \le x \le \pi$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh(x)\, e^{inx}\,dx = \frac{1}{4\pi}\int_{-\pi}^{\pi} (e^{inx+x} + e^{inx-x})\,dx$$

$$= \frac{1}{4\pi}\left(\frac{e^{inx+x}}{in+1} + \frac{e^{inx-x}}{in-1}\right)\Bigg|_{-\pi}^{\pi} = \frac{1}{4\pi}\,\frac{in(e^x+e^{-x}) - (e^x-e^{-x})}{-(n^2+1)}\,e^{inx}\Bigg|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\,\frac{1}{n^2+1}\left[(e^{in\pi}+e^{-in\pi})\sinh(\pi) - in(e^{in\pi}-e^{-in\pi})\cosh(\pi)\right]$$

$$= \frac{1}{\pi}\,\frac{1}{n^2+1}\left[\cos(n\pi)\sinh(\pi) + n\,\overset{0}{\cancel{\sin(n\pi)}}\cosh(\pi)\right] = \frac{(-)^n \sinh(\pi)}{\pi(n^2+1)}$$

$$\Rightarrow \boxed{\cosh(x) = \frac{\sinh(\pi)}{\pi}\sum_n \frac{(-)^n}{n^2+1}\,e^{inx}}$$

For $x=0$, $\cosh(x) = 1$ and $e^{inx} = 1$

$$\Rightarrow 1 = \frac{\sinh(\pi)}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-)^n}{n^2+1} = \frac{\sinh(\pi)}{\pi}\left[1 + 2\sum_{n=1}^{\infty}\frac{(-)^n}{n^2+1}\right]$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{(-)^n}{n^2+1} = \frac{1}{2}\left[\frac{\pi}{\sinh(\pi)} - 1\right]}$$

[Text 4.19] ⑨ Square wave $\rightarrow a_n = 0,\ b_n = \frac{4}{\pi n} \Rightarrow f(t) = \frac{4}{\pi}\sum_{n\,odd}\frac{\sin(n\omega t)}{n}$

$$\int_{-T/2}^{T/2} |f(t)|^2\,dt = \int_{-T/2}^{T/2} dt = T \quad \text{and} \quad \sum_n |b_n|^2 = \frac{16}{\pi^2}\sum_{n\,odd}\frac{1}{n^2} = \frac{16}{\pi^2}\,\frac{\pi^2}{8} = 2$$

(9) [continued]

$$\Rightarrow \frac{1}{T}\int_{-T/2}^{T/2} |f|^2 dt = \frac{1}{2}\sum |b_n|^2 \rightarrow \underline{\text{Parseval thm satisfied}}$$

$$\omega = 2\pi/T \Rightarrow 8\pi/T = 4\omega \rightarrow \omega_1 \text{ and } \omega_3 < 8\pi/T$$

$$\rightarrow |b_1|^2 + |b_3|^2 = 16/\pi^2 (1 + 1/9) = 160/9\pi^2 = \underline{1.801}$$

Complete sum $\longrightarrow 2 \Rightarrow \underline{|b_1|^2 + |b_3|^2 = 0.9 \sum |b_n|^2}$

(10) (a) $f(x) = (2h/L)(L/2 - |x|)$ for $-L/2 \le x \le L/2 \rightarrow$ symmetric $\rightarrow b_n = 0$

$$a_0 = \left(\frac{4}{L}\right)\left(\frac{2h}{L}\right)\int_0^{L/2}\left(\frac{L}{2} - x\right)dx = \frac{8h}{L^2}\left[\left(\frac{L}{2}\right)^2 - \frac{1}{2}\left(\frac{L}{2}\right)^2\right] = h$$

$$a_n = \frac{8h}{L^2}\int_0^{L/2}\left(\frac{L}{2} - x\right)\cos\left(\frac{2\pi n x}{L}\right)dx = \frac{8h}{L^2}\left(\frac{L}{2\pi n}\right)\left[\cancelto{0}{\left(\frac{L}{2} - x\right)\sin\left(\frac{2\pi n x}{L}\right)\Big|_0^{L/2}} + \int_0^{L/2}\sin\left(\frac{2\pi n x}{L}\right)dx\right]$$

$$= -\frac{8h}{L^2}\left(\frac{L}{2\pi n}\right)^2 \cos\left(\frac{2\pi n x}{L}\right)\Big|_0^{L/2} = \frac{2h}{(n\pi)^2}\left[1 - \cos(n\pi)\right] = \frac{4h}{(n\pi)^2} \quad n \text{ odd}$$

$$\Rightarrow \boxed{f(x) = \frac{h}{2} + \frac{4h}{\pi^2}\sum_{n\text{ odd}} \frac{\cos(2\pi n x/L)}{n^2}}$$

(b) Integrating gives

$$\int_{-L/2}^{L/2} |f(x)|^2 dx = 2\left(\frac{2h}{L}\right)^2 \int_0^{L/2}\left(\frac{L^2}{4} - Lx + x^2\right)dx = \frac{8h^2}{L^2}\left[\frac{L^3}{8} - \frac{L}{2}\left(\frac{L}{2}\right)^2 + \frac{1}{3}\left(\frac{L}{2}\right)^3\right]$$

$$\rightarrow \boxed{\int_{-L/2}^{L/2} |f(x)|^2 dx = \frac{h^2 L}{3}}$$

(c) Parseval $\rightarrow \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_n |a_n|^2 = \frac{1}{L}\int |f(x)|^2 dx$

$$\Rightarrow \frac{h^2}{4} + \frac{1}{2}\frac{16h^2}{\pi^4}\sum_{n\text{ odd}}\frac{1}{n^4} = \frac{h^2}{3} \Rightarrow \boxed{\sum_{n\text{ odd}}\frac{1}{n^4} = \frac{\pi^4}{8}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi^4}{96}}$$