Set 6 — Solutions —

(1) (a) $f(t) = e^{-at}$

sine series $\rightarrow \tilde{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \sin(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \frac{1}{2i} \int_0^\infty (e^{-at+i\omega t} - e^{-at-i\omega t})\, dt$

$= -\frac{i}{\sqrt{2\pi}} \left[\frac{1}{-a-i\omega} - \frac{1}{-a+i\omega} \right] = -\frac{i}{\sqrt{2\pi}} \frac{2i\omega}{a^2+\omega^2} \Rightarrow \boxed{\tilde{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \frac{\omega}{a^2+\omega^2}}$

cosine series $\rightarrow \tilde{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_0^\infty (e^{-at+i\omega t} + e^{-at-i\omega t})\, dt$

$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{a+i\omega} + \frac{1}{a-i\omega} \right] = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2+\omega^2} \Rightarrow \boxed{\tilde{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+\omega^2}}$

(b) Inverse transforms:

$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_s(\omega) \sin(\omega t)\, d\omega \longrightarrow \boxed{f(t) = \frac{2}{\pi} \int_0^\infty \frac{\omega \sin(\omega t)}{a^2+\omega^2}\, d\omega}$

$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \tilde{f}_c(\omega) \cos(\omega t)\, d\omega \longrightarrow \boxed{f(t) = \frac{2}{\pi} a \int_0^\infty \frac{\cos(\omega t)}{a^2+\omega^2}\, d\omega}$

[Text 5.2] (2) (a) Define $x' = x+a$ so that $x = x'-a$ and $dx = dx'$

$\rightarrow \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x'-a) e^{-ik(x'-a)}\, dx'$

$= \frac{1}{\sqrt{2\pi}} e^{ika} \int_{-\infty}^\infty f(x') e^{-ikx'}\, dx'$ since $f(x'-a) = f(x')$

$\Rightarrow \tilde{f}(k) = e^{ika} \tilde{f}(k) \Rightarrow \tilde{f}(k) = 0$ if $e^{ika} \neq 1$

$\Rightarrow \boxed{\tilde{f}(k) = 0 \text{ if } ka \neq 2\pi n}$

(b) Let $F(\omega)$ be the Fourier transf of $t f(t)$

$\rightarrow F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty t f(t) e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t)\, i \frac{\partial}{\partial\omega} (e^{-i\omega t})\, dt$

$= i \frac{d}{d\omega} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t) e^{-i\omega t}\, dt \right] \Rightarrow \boxed{F(\omega) = i \frac{d\tilde{f}}{d\omega}}$

(2) (c) Let $F(\omega)$ be the Fourier transf. of $f(mt+c)$

$$\rightarrow F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(mt+c)\, e^{-i\omega t}\, dt \rightarrow \text{define } u = mt+c \Rightarrow t = \frac{u-c}{m}$$

$$dt = \frac{du}{m} \Rightarrow F(\omega) = \frac{1}{\sqrt{2\pi}} \frac{e^{i\omega c/m}}{m} \int_{-\infty}^{\infty} f(u)\, e^{-\omega u/m}\, du$$

$$\Rightarrow \boxed{F(\omega) = \left[e^{i\omega c/m}/m\right] \tilde{f}(\omega/m)}$$

(3) (a) $-D\, d^2f/dx^2 + K^2 D\, f(x) = Q\, \delta(x)$

The Fourier transf. of d^2f/dx^2 is $-k^2 \tilde{f}(k)$ and the Fourier transf of $\delta(x)$ is $1/\sqrt{2\pi}$

$$\Rightarrow D k^2 \tilde{f}(k) + K^2 D \tilde{f}(k) = \frac{Q}{\sqrt{2\pi}} \Rightarrow \boxed{\tilde{f}(k) = \frac{Q}{\sqrt{2\pi}\, D} \frac{1}{k^2+K^2}}$$

(b) $$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{f}(k) \cos(kx)\, dk = \sqrt{\frac{2}{\pi}} \frac{Q}{\sqrt{2\pi}\, D} \int_0^{\infty} \frac{\cos(kx)}{k^2+K^2}\, dk$$

Now $$\int_0^{\infty} \frac{\cos(kx)}{k^2+K^2}\, dk = \frac{\pi}{2K} e^{-Kx} \Rightarrow \boxed{f(x) = \frac{Q}{2KD} e^{-K|x|}}$$ (symmetric)

Text 5.1] (4) (a) $f(t) = e^{-|t|}$ → symmetric → use cosine transform

$$\rightarrow \tilde{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-t} \cos(\omega t)\, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \left(e^{-t+i\omega t} + e^{-t-i\omega t}\right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1+i\omega} + \frac{1}{1-i\omega} \right) = \frac{1}{\sqrt{2\pi}} \frac{2}{1+\omega^2} \Rightarrow \tilde{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\omega^2}$$

$$\Rightarrow e^{-|t|} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \tilde{f}(\omega) \cos(\omega t)\, d\omega \rightarrow \boxed{\frac{\pi}{2} e^{-|t|} = \int_0^{\infty} \frac{\cos(\omega t)}{1+\omega^2}\, d\omega}$$

4 b $\underline{\int_{-\infty}^{\infty} |f(t)|^2 dt} = 2\int_0^{\infty} e^{-2t} dt = -e^{-2t}\Big|_0^{\infty} = \underline{1}$

$\int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega = 2\left(\frac{2}{\pi}\right)\int_0^{\infty} \frac{d\omega}{(1+\omega^2)^2}$ → define $\omega = \tan\theta$

$\Rightarrow d\omega = \sec^2\theta\, d\theta = d\theta/\cos^2\theta \qquad (1+\omega^2)^{-1} = (1+\tan^2\theta)^{-1} = \cos^2\theta$

$\Rightarrow \underline{\int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega} = \frac{4}{\pi}\int_0^{\pi/2} \frac{\cos^4\theta}{\cos^2\theta} d\theta = \frac{2}{\pi}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta = \underline{1}$

→ Parseval thm satisfied

5 a $\partial^2 q/\partial x^2 = \partial q/\partial t$

Fourier transf of $\partial^2 q/\partial x^2 = -k^2 Q \Rightarrow -k^2 Q = \partial Q/\partial t$

$\rightarrow dQ/Q = -k^2 dt \rightarrow \underline{Q = A e^{-k^2 t}}$

b $q(x, t=0) = S\,\delta(x)$

$\Rightarrow Q(k, t=0) = \frac{S}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx = \frac{S}{\sqrt{2\pi}} \Rightarrow \boxed{A = \frac{S}{\sqrt{2\pi}}}$

c $q = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} Q e^{ikx} dk = \frac{S}{2\pi}\int_{-\infty}^{\infty} e^{-k^2 t + ikx} dk$

define $u^2 = k^2 t - ikx - x^2/4t = (k - ix/2t)^2 t$

$\Rightarrow du = \sqrt{t}\, dk$

$\Rightarrow q = \frac{S}{2\pi}\frac{1}{\sqrt{t}} e^{-x^2/4t}\int_{-\infty}^{\infty} e^{-u^2} du = \frac{S}{2\pi\sqrt{t}} e^{-x^2/4t}\sqrt{\pi}$

$\Rightarrow \boxed{q(x,t) = \frac{S}{\sqrt{4\pi t}} e^{-x^2/4t}}$

[Text 5.7] (6) $f(t) = 1, \ |t| < 1 \rightarrow$ symmetric $\rightarrow$ cosine transf

$$\rightarrow \tilde{f}(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \int_0^1 \cos(\omega t)\, dt = \sqrt{\frac{2}{\pi}} \left. \frac{\sin(\omega t)}{\omega} \right|_0^1$$

$$\Rightarrow \boxed{\tilde{f}(\omega) = \sqrt{2/\pi}\ \sin\omega/\omega}$$

Convolution integral

$$\lambda(x) = \int_{-\infty}^{\infty} f(t) f(x-t)\, dt = \int_{-1}^{1} f(x-t)\, dt$$

Now $f(x-t) = 0$ for $|x-t| > 1 \Rightarrow |x-t| < 1$

For $x > t$, $x-t < 1 \Rightarrow t > x-1 \rightarrow x < 2$

For $x < t$, $t-x < 1 \Rightarrow t < x+1 \rightarrow x > -2$

$$\Rightarrow \underline{\text{for } x > 0}, \ \lambda(x) = \int_{x-1}^{1} dt = 2 - x \ ; \ \underline{\text{for } x < 0}, \ \lambda(x) = \int_{-1}^{x+1} dt = 2 + x$$

$$\Rightarrow \boxed{\lambda(x) = 2 - |x|} \quad \text{and} \quad \underline{|x| < 2}$$

From the convolution thm, $\tilde{\lambda}(\omega) = \sqrt{2\pi}\,[\tilde{f}(\omega)]^2 = 2\sqrt{2/\pi}\ \sin^2\omega/\omega^2$

From Parseval's thm

$$\int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(t)|^2 dt = 2 \Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{\sin^2\omega}{\omega^2}\, d\omega = \pi}$$

$$\int_{-\infty}^{\infty} |\tilde{\lambda}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |\lambda(x)|^2 dx = 2\int_0^2 (4 - 4x + x^2)\, dx = 2\left(8 - 2(4) + \frac{8}{3}\right) = \frac{16}{3}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{\sin^4\omega}{\omega^4}\, d\omega = \frac{2}{3}\pi}$$

[Text 5.14] (7) Let $f(t) = e^{-at} \sin(at)$ for $0 \le t \le \infty$

$$\Rightarrow \underline{\tilde{f}(\omega)} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-at} \sin(at)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-at-iat}\ \frac{e^{iat} - e^{-iat}}{2i}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-i}{2} \left[\frac{1}{a + i(\omega - a)} - \frac{1}{a + i(\omega + a)} \right] = \frac{1}{\sqrt{2\pi}} \ \underline{\frac{a}{(a + i\omega)^2 + a^2}}$$

⑦ [continued]

Parseval's thm

$$\to \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega \Rightarrow \int_0^{\infty} e^{-2at} \sin^2 at\, dt = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2 d\omega$$

Now

$$|\tilde{f}(\omega)|^2 = \frac{a^2}{2\pi}\left[\frac{1}{(a+i\omega)^2+a^2}\right]\left[\frac{1}{(a-i\omega)^2+a^2}\right]$$

$$= \frac{a^2}{2\pi}\left(\frac{1}{2a^2-\omega^2+2ia\omega}\right)\left(\frac{1}{2a^2-\omega^2-2ia\omega}\right) = \frac{a^2}{2\pi}\,\frac{1}{(2a^2-\omega^2)^2+4a^2\omega^2}$$

$$= \frac{a^2}{2\pi}\,\frac{1}{4a^4+\omega^4} \Rightarrow \boxed{\int_0^{\infty} e^{-2at} \sin^2 at\, dt = \frac{a^2}{\pi}\int_0^{\infty}\frac{d\omega}{4a^4+\omega^4}}$$

⑧ ⓐ $f(x) = 1 - |x|/2$ for $|x| \le 2$ → symmetric → cosine transform

$$\to \tilde{f}(k) = \sqrt{\frac{2}{\pi}}\int_0^2 \left(1-\frac{x}{2}\right)\cos(kx)\,dx = \sqrt{\frac{2}{\pi}}\,\cancel{\frac{1-x/2}{k}\sin kx}\Big|_0^2 \overset{0}{} + \sqrt{\frac{2}{\pi}}\frac{1}{2k}\int_0^2 \sin(kx)\,dx$$

$$= -\sqrt{\frac{2}{\pi}}\,\frac{1}{2k^2}\cos(kx)\Big|_0^2 = \sqrt{\frac{2}{\pi}}\,\frac{1}{2k^2}\left[1-\cos(2k)\right] = \sqrt{\frac{2}{\pi}}\,\frac{\sin^2 k}{k^2}$$

$$\Rightarrow \boxed{\tilde{f}(k) = \sqrt{2/\pi}\,(\sin k/k)^2}$$

ⓑ Parseval's thm

$$\to \int_{-\infty}^{\infty}|f(x)|^2 dx = \int_{-\infty}^{\infty}|\tilde{f}(k)|^2 dk \Rightarrow 2\int_0^2\left(1-\frac{x}{2}\right)^2 dx = \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin^4 k}{k^4}\,dk$$

$$\Rightarrow \int_{-\infty}^{\infty}\left(\frac{\sin k}{k}\right)^4 dk = \pi\int_0^2\left(1-x+\frac{x^2}{4}\right)dx = \pi\left(2-\frac{1}{2}2^2+\frac{2^3}{12}\right) = \frac{2}{3}\pi$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty}\left(\frac{\sin k}{k}\right)^4 dk = \frac{2}{3}\pi}$$

(9) $F(\vec{k}) = \left[(2\pi)^{3/2}k^2\right]^{-1}$

$\rightarrow f(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \int F(\vec{k}) e^{i\vec{k}\cdot\vec{r}} d^3k$

spherical coords with $\hat{z}$ in the direction of $\vec{r}$

$\Rightarrow \vec{k}\cdot\vec{r} = kr\cos\theta$ and $d^3k = k^2 dk\, d(\cos\theta)\, d\phi$

$\Rightarrow f(\vec{r}) = \frac{1}{(2\pi)^3} \int_0^\infty k^2 dk \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \frac{e^{ikr\cos\theta}}{k^2}$

$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dk \left. \frac{e^{ikru}}{ikr} \right|_{-1}^{1} = \frac{4\pi}{(2\pi)^3} \int_0^\infty \frac{\sin kr}{kr} dk$

$= \frac{4\pi}{(2\pi)^3} \frac{1}{r} \int_0^\infty \frac{\sin u}{u} du = \frac{4\pi}{(2\pi)^3} \frac{1}{r} \frac{\pi}{2} \rightarrow \boxed{f(\vec{r}) = \frac{1}{4\pi r}}$