t 7 — Solutions —

① (a) $z = x+iy \Rightarrow \boxed{1/z = (x+iy)^{-1} = (x-iy)/(x^2+y^2)}$

In polar form $\boxed{z^{-1} = (Re^{i\theta})^{-1} = e^{-i\theta}/R}$

(b) $z = \pm i = e^{\pm i\pi/2} \Rightarrow \sqrt{z} = \pm e^{\pm i\pi/4} = \pm\left[\cos(\frac{\pi}{4}) \pm i\sin(\frac{\pi}{4})\right]$

$\Rightarrow \boxed{\sqrt{z} = \pm(1 \pm i)/\sqrt{2}}$

② (a) $\sin(z) = -i/2\left[e^{i(x+iy)} - e^{-i(x+iy)}\right] = -i/2(e^{-y}e^{ix} - e^{y}e^{-ix})$

$= -\frac{i}{2}e^{-y}(\cos x + i\sin x) + \frac{i}{2}e^{y}(\cos x - i\sin x)$

$= 1/2\sin x(e^{-y}+e^{y}) + i/2\cos x(e^{y}-e^{-y}) = \sin x\cosh y + i\cos x\sinh y$

$\cos(z) = 1/2[e^{i(x+iy)} + e^{-i(x+iy)}] = 1/2(e^{-y}e^{ix} + e^{y}e^{-ix})$

$= \frac{1}{2}e^{-y}(\cos x + i\sin x) + \frac{1}{2}e^{y}(\cos x - i\sin x)$

$= 1/2\cos x(e^{-y}+e^{y}) + i/2\sin x(e^{-y}-e^{y}) = \cos x\cosh y - i\sin x\sinh y$

(b) $|\sin z|^2 = (\sin z)(\sin z)^*$

$= (\sin x\cosh y + i\cos x\sinh y)(\sin x\cosh y - i\cos x\sinh y)$

$= \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$

$= \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y = \sin^2 x + \sinh^2 y$

$|\cos z|^2 = (\cos x\cosh y - i\sin x\sinh y)(\cos x\cosh y + i\sin x\sinh y)$

$= \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y$

$= \cos^2 x(1+\sinh^2 y) + (1-\cos^2 x)\sinh^2 y = \cos^2 x + \sinh^2 y$

(c) $\sin z = 0 \Rightarrow \sin x = \sinh y = 0 \Rightarrow \boxed{x = n\pi,\ y=0}$

$\cos z = 0 \Rightarrow \cos x = \sinh y = 0 \Rightarrow \boxed{x = (n+1/2)\pi,\ y=0}$

Note that $\cosh y$ is never zero and $\sin x$ and $\cos x$ cannot both be zero for the same x

(3) (a) Let $u = \sin^{-1}(z) \Rightarrow z = \sin u = -i/2\,(e^{iu} - e^{-iu})$

$\Rightarrow e^{iu} - e^{-iu} = 2iz \Rightarrow (e^{iu})^2 - 2iz(e^{iu}) - 1 = 0$

$\Rightarrow e^{iu} = iz \pm \sqrt{-z^2+1} \Rightarrow iu = \ln[iz \pm \sqrt{1-z^2}]$

$\Rightarrow \boxed{\sin^{-1}(z) = u = -i\ln[iz \pm \sqrt{1-z^2}]}$

(b) Let $u = \tan^{-1}(z) \Rightarrow z = \tan u = -i(e^{iu} - e^{-iu})/(e^{iu} + e^{-iu})$

$= -i(e^{2iu} - 1)/(e^{2iu} + 1) \Rightarrow e^{2iu}(z+i) = i - z$

$\Rightarrow e^{2iu} = (i-z)/(i+z) \Rightarrow \tan^{-1} z = u = -i/2\,\ln[(i-z)/(i+z)]$

$\rightarrow \boxed{\tan^{-1} z = i/2\,\ln[(i+z)/(i-z)]}$

(4) (a) $f(z) = u + iv \Rightarrow \partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ if analytic

$f^*(z) = u - iv \Rightarrow \partial u/\partial x = -\partial v/\partial y$ and $\partial u/\partial y = \partial v/\partial x$ if analytic

$\Rightarrow \partial v/\partial y = -\partial v/\partial y$ and $\partial v/\partial x = -\partial v/\partial x \Rightarrow \partial v/\partial y = \partial v/\partial x = 0$

$\Rightarrow$ v constant $\Rightarrow \partial u/\partial x = \partial u/\partial y = 0 \Rightarrow$ u constant

(b) $f(z) = u + iv \Rightarrow \partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ if analytic

$f^*(z^*) = u(x,-y) - iv(x,-y)$

$\rightarrow$ analytic if $\partial u/\partial x = -\partial v/\partial(-y) = +\partial v/\partial y$

and if $\partial u/\partial(-y) = +\partial v/\partial x \Rightarrow \partial u/\partial y = -\partial v/\partial x$

$\rightarrow$ CR relations for $f^*(z^*)$ same as for $f(z)$

$\Rightarrow$ $f^*(z^*)$ analytic if $f(z)$ is

[Text 14.1] (5) $v = (y\cos y + x\sin y)e^x \rightarrow \partial v/\partial x = [y\cos y + (x+1)\sin y]e^x$

and $\partial v/\partial y = [(x+1)\cos y - y\sin y]e^x$

$\partial u/\partial x = \partial v/\partial y \Rightarrow u = (x\cos y - y\sin y)e^x + \alpha(y)$

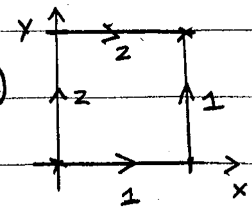
$\Rightarrow \partial u/\partial y = -[(x+1)\sin y + y\cos y]e^x + d\alpha/dy = -\partial v/\partial x$

$\Rightarrow d\alpha/dy = 0 \Rightarrow \alpha = c$ constant $\Rightarrow u = (x\cos y - y\sin y)e^x + c$

$\Rightarrow f(z) = [(x\cos y - y\sin y) + i(y\cos y + x\sin y)]e^x + c$

$= (x+iy)[\cos y + i\sin y]e^x + c \rightarrow \boxed{f(z) = z e^z + c}$

(6) (a)



For $f(z) = x+iy$,

$$\int_{\text{path 1}} f(z)\,dz = \int_0^1 f(x, y=0)\,dx + \int_0^1 f(x=1, y)\, i\,dy$$

$$= \int_0^1 x\,dx + \int_0^1 (1+iy)\, i\,dy = 1/2 + i(1+\tfrac{i}{2}) = \underline{\underline{i}}$$

$$\int_{\text{path 2}} f(z)\,dz = \int_0^1 f(x=0,y)\, i\,dy + \int_0^1 f(x,y=1)\,dx = \int_0^1 (iy)\, i\,dy + \int_0^1 (x+i)\,dx$$

$$= -1/2 + 1/2 + i = \underline{\underline{i}} \rightarrow \underline{\underline{\text{same result}}}$$

(b) For $f(z) = z^* = x - iy$,

$$\int_{\text{path 1}} f(z)\,dz = \int_0^1 x\,dx + \int_0^1 (1-iy)\, i\,dy = \tfrac{1}{2} + i(1 - 1/2) = \underline{\underline{1+i}}$$

$$\int_{\text{path 2}} f(z)\,dz = \int_0^1 (-iy)\, i\,dy + \int_0^1 (x-i)\,dx = \tfrac{1}{2} + \tfrac{1}{2} - i = \underline{\underline{1-i}}$$

$\rightarrow$ <u>different results for the non-analytic fcn</u>

Text 14.7] (7) (i) $f(z) = z^2 = (x+iy)^2 = \underline{(x^2 - y^2) + i(2xy)}$

(ii) $f(z) = e^z = e^{x+iy} = \underline{e^x \cos y + i(e^x \sin y)}$

(iii) $f(z) = \cosh(\pi z) = 1/2\,(e^{\pi z} + e^{-\pi z})$

$$= 1/2\, e^{\pi x}(\cos \pi y + i \sin \pi y) + 1/2\, e^{-\pi x}(\cos \pi y - i \sin \pi y)$$

$$= 1/2\,(e^{\pi x} + e^{-\pi x}) \cos \pi y + i\, 1/2\,(e^{\pi x} - e^{-\pi x}) \sin \pi y$$

$$= \underline{\cosh(\pi x) \cos(\pi y) + i\,[\sinh(\pi x) \sin(\pi y)]}$$

Now $\underline{\phi(x,y) = 0 \text{ for } x = 0 \text{ or } y = 0}$

$\rightarrow$ eliminates $x^2 - y^2$, $e^x \cos y$, $e^x \sin y$, and $\cosh(\pi x) \cos(\pi y)$

$\rightarrow$ fcns $2xy$ and $\sinh(\pi x) \sin(\pi y)$ satisfy both conditions

$\Rightarrow \phi(x,y) = Axy + B \sinh(\pi x) \sin(\pi y)$

Now $\phi(x,y) = x$ for $y = 1 \Rightarrow \underline{\underline{A = 1}}$

and $\phi(x,y) = y + \sin(\pi y)$ for $x = 1 \Rightarrow B = 1/\sinh(\pi)$

$$\Rightarrow \boxed{\phi(x,y) = xy + \sinh(\pi x) \sin(\pi y) / \sinh(\pi)}$$

(8) $f(z) = (e^z - 1)^{-1}$

Taylor series for $e^z \to e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots$

$\Rightarrow e^z - 1 = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots = z(1 + \frac{1}{2}z + \frac{1}{6}z^2 + \cdots)$

Now define u so that $e^z - 1 = z(1+u)$

$\Rightarrow (e^z-1)^{-1} = \frac{1}{z}(1+u)^{-1} \cong \frac{1}{z}(1 - u + u^2 + \cdots)$

and $u = \frac{1}{2}z + \frac{1}{6}z^2 + \cdots$

$\Rightarrow (e^z-1)^{-1} \cong \frac{1}{z}(1 - \frac{1}{2}z - \frac{1}{6}z^2 + \frac{1}{4}z^2)$ up to order z^2

$\rightarrow \boxed{(e^z-1)^{-1} \cong \frac{1}{z} - \frac{1}{2} + z/12}$ up to order z

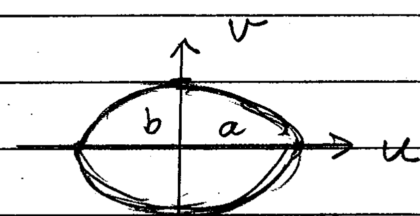
(9) (a) For $w = z + 1/z$, $u + iv = (x+iy) + (x-iy)/(x^2+y^2)$

$\Rightarrow u = x[1 + 1/(x^2+y^2)]$ and $v = y[1 - 1/(x^2+y^2)]$

$\Rightarrow u = x(1 + 1/R^2)$ and $v = y(1 - 1/R^2)$ for $x^2+y^2 = R^2$

$$\Rightarrow \frac{u^2}{(1+1/R^2)^2} + \frac{v^2}{(1-1/R^2)^2} = x^2+y^2 = R^2$$

$$\Rightarrow \frac{u^2}{(R+1/R)^2} + \frac{v^2}{(R-1/R)^2} = 1 \rightarrow \text{ellipse}$$

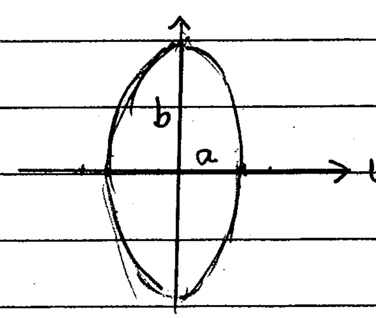


$\rightarrow \boxed{a = R + 1/R \text{ (semi-major)}, \quad b = R - 1/R \text{ (semi-minor)}}$

For $w = z - 1/z$,

$u = x(1 - 1/R^2)$ and $v = y(1 + 1/R^2)$

$$\Rightarrow \frac{u^2}{(R-1/R)^2} + \frac{v^2}{(R+1/R)^2} = 1 \rightarrow \text{ellipse}$$



$\rightarrow \boxed{a = R - 1/R, \quad b = R + 1/R}$

(b) When $R = 1$, $R - 1/R = 0$ and $R + 1/R = 2$

$\rightarrow$ ellipses become straight lines of length 2

10 (a) $e^z = (a-w)/(a+w) \Rightarrow (a+w)e^z = a-w$

$\Rightarrow w(e^z+1) = a(1-e^z) \Rightarrow w = a(1-e^z)/(1+e^z)$

On the z-space real axis, $z = x \Rightarrow w = a(1-e^x)/(1+e^x)$

$\rightarrow \underline{w} = -a(e^{x/2}-e^{-x/2})/(e^{x/2}+e^{-x/2}) = \underline{-a\tanh(x/2)}$

(b) On the z-space imag axis, $z = iy$

$\Rightarrow w = a(1-e^{iy})/(1+e^{iy}) = -a(e^{iy/2}-e^{-iy/2})/(e^{iy/2}+e^{-iy/2})$

$\rightarrow \boxed{w = -ia\tan(\frac{y}{2})}$